Faculty of Science, Technology, Engineering and Mathematics
M208 Pure Mathematics

Additional exercises for Book A

Contents

WEB 07700 9
2.1

Additional exercises for Unit A1

Section 1

Additional Exercise A1

Determine the equation of the line through each of the following pairs of points, showing that the equation can be written in the form $ax + by = c$, for some real numbers a, b and c, where a and b are not both zero.

(a) $(-2, -4)$ and $(1, 6)$ (b) $(0, 0)$ and $(7, 3)$

Additional Exercise A2

Determine the values of k for which the lines

$$3x + 4y + 7 = 0 \quad \text{and} \quad 2x + ky = 3$$

are (a) parallel, (b) perpendicular.

Additional Exercise A3

Determine the equation of the circle with centre $(2, 1)$ and radius 3.

Additional Exercise A4

Sketch, on a single diagram, the line and circle with the following equations:

$$2x - y - 2 = 0, \quad (x - 3)^2 + (y - \tfrac{1}{2})^2 = 9.$$

You do not need to find the points of intersection of the line and circle.

Additional Exercise A5

Determine the distance between the centre of the circle and the y-intercept of the line in Additional Exercise A4.

Additional Exercise A6

Determine the distance between the points $(1, -2, 3)$ and $(-2, 3, -1)$ in $\mathbb{R}^3$.

Section 2

Additional Exercise A7

Which of the following statements are true?

(a) $0 \in \mathbb{N}$ (b) $0 \in \mathbb{Q}$ (c) $37 \in \mathbb{Z}$

(d) $-0.6 \notin \mathbb{R}$ (e) $20 \in \{4, 8, 12, 16\}$

(f) $4, 8, 16 \in \{4, 8, 12, 16\}$ (g) $\{0\} \in \varnothing$

(h) $\{1, 2\} \in \{\{2, 3\}, \{3, 1\}, \{2, 1\}\}$

(i) $(1, 2) \in \{(2, 3), (2, 1)\}$

Additional Exercise A8

List the elements of the following sets.

(a) $\{n \in \mathbb{N} : 2 < n < 7\}$

(b) $\{x \in \mathbb{R} : x^2 + 5x + 4 = 0\}$

(c) $\{n \in \mathbb{N} : n^2 = 25\}$

Additional Exercise A9

Use set notation to specify each of the following sets.

(a) The set of integers greater than -20 and less than -3.

(b) The set of non-zero integers which are multiples of 3.

(c) The set of all real numbers greater than 15.

(d) The interval $(-\pi, 2\sqrt{2}]$.

(e) The interval $[71, \infty)$.

Additional Exercise A10

Sketch the following sets in $\mathbb{R}^2$.

(a) $\{(x, y) \in \mathbb{R}^2 : y = 4 - 3x\}$

(b) $\{(x, y) \in \mathbb{R}^2 : (x + 1)^2 + (y - 3)^2 = 9\}$

(c) $\{(x, \tfrac{1}{2}x^2) : x \in \mathbb{R}\}$

(d) $\{(x, y) \in \mathbb{R}^2 : y^2 = 2x\}$

Additional Exercise A11

Sketch the following sets in $\mathbb{R}^2$.

(a) $\{(x,y) \in \mathbb{R}^2 : y < 4 - 3x\}$

(b) $\{(x,y) \in \mathbb{R}^2 : (x+1)^2 + (y-3)^2 > 9\}$

(c) $\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2,\ -1 < y < 1\}$

Additional Exercise A12

For each of the sets A and B below, determine whether $A \subseteq B$.

(a) $A = \{(0,0), (0,6), (-4,6)\}$ and $B = \{(x,y) \in \mathbb{R}^2 : (x+2)^2 + (y-3)^2 = 13\}$.

(b) $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$ and $B = \{(x,y) \in \mathbb{R}^2 : y < 4 - 8x\}$.

Additional Exercise A13

Show that A is a proper subset of B, where

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + 4y^2 < 1\} \text{ and}$$
$$B = \{(x,y) \in \mathbb{R}^2 : y < \tfrac{1}{2}\}.$$

Additional Exercise A14

For each of the sets A and B below, determine whether $A = B$.

(a) $A = \{1, -1, 2\}$ and $B = \{x \in \mathbb{R} : x^3 - 2x^2 - x + 2 = 0\}$.

(b) $A = \{(\cos t, 2\sin t) : t \in [0, 2\pi]\}$ and $B = \{(x,y) \in \mathbb{R}^2 : 4x^2 + y^2 = 4\}$.

Hint: Try dividing $4x^2 + y^2 = 4$ through by 4.

(c) $A = \{x \in \mathbb{R} : x = \dfrac{p}{q}$, where $p, q \in \mathbb{N}\}$ and $B = \mathbb{Q}$.

Additional Exercise A15

For each of the sets A and B below, find $A \cup B$, $A \cap B$ and $A - B$.

(a) $A = \{0, 2, 4\}$ and $B = \{4, 5, 6\}$.

(b) $A = (-5, 3]$ and $B = [2, 17]$.

(c) $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

Additional Exercise A16

Sketch the sets A, B, $A \cup B$, $A \cap B$, $A - B$ and $B - A$, where

$$A = \{(x,y) \in \mathbb{R}^2 : y < 2 - 2x\} \text{ and}$$
$$B = \{(x,y) \in \mathbb{R}^2 : (x+1)^2 + (y-2)^2 > 4\}.$$

Additional Exercise A17 Challenging

Sketch the sets $A \cap B \cap C$ and $D = ((A \cap B) \cup (B \cap C)) - (A \cap C)$, where

$$A = \{(x,y) \in \mathbb{R}^2 : x + y \leq 0\},$$
$$B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \text{ and}$$
$$C = \{(x,y) \in \mathbb{R}^2 : x - y < 0\}.$$

(Take care with the points on the boundaries!)

Section 3

Additional Exercise A18

For each of the following transformations $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, state whether f is a translation, reflection or rotation of the plane.

(a) $f(x,y) = (y, -x)$

(b) $f(x,y) = (x-2, y+1)$

Additional Exercise A19

Draw a diagram showing the image of T, the triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$, under each of the functions f of Additional Exercise A18.

Additional Exercise A20

For each of the following functions, find its image set and determine whether it is onto.

(a) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x,y) \longmapsto (-y, x)$

(b) $f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto 7 - 3x$

(c) $f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto x^2 - 4x + 3$

(d) $f : [0,1] \longrightarrow \mathbb{R}$
$x \longmapsto 2x + 3$

Additional Exercise A21

Determine which of the functions in Additional Exercise A20 are one-to-one.

Additional Exercise A22

Determine which of the functions in Additional Exercise A20 has an inverse, and find the inverse f^{-1} for each one which does.

Additional Exercise A23

Determine the composite function $f \circ g$ for each of the following pairs of functions f and g.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto 7 - 3x$

and

$g : \mathbb{R} - \{2, -2\} \longrightarrow \mathbb{R}$
$x \longmapsto \dfrac{1}{x^2 - 4}.$

(b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (-y, x)$

and

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (y, x).$

Additional Exercise A24

Let f and g be the functions

$$f : \{0, 1, 2, 3\} \longrightarrow \{0, 1, 4, 9\}$$
$$x \longmapsto x^2$$

and

$$g : \{0, 1, 4, 9\} \longrightarrow \{0, 5, 8, 9\}$$
$$x \longmapsto 9 - x.$$

(a) Find $g \circ f$, f^{-1}, g^{-1} and $f^{-1} \circ g^{-1}$, specifying the domain and codomain in each case.

(b) Use Strategy A2 to show that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$.

Additional Exercise A25 Challenging

Let f and g be functions with inverses f^{-1} and g^{-1}, respectively, such that

$$f : A \longrightarrow B \quad \text{and} \quad g : B \longrightarrow C.$$

Use Strategy A2 to show that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$.

Section 4

Additional Exercise A26

Let $\mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ be two vectors in $\mathbb{R}^3$. Determine $\mathbf{p} + \mathbf{q}$, $\mathbf{p} - \mathbf{q}$ and $2\mathbf{p} - 3\mathbf{q}$.

Additional Exercise A27

Let $\mathbf{u}$ and $\mathbf{v}$ be the position vectors of the points $(1, 1)$ and $(2, 1)$, respectively.

(a) Determine the vectors $\mathbf{u} + 2\mathbf{v}$, $-\mathbf{u}$, $-\mathbf{u} + 3\mathbf{v}$ and $\mathbf{u} - 3\mathbf{v}$.

(b) Sketch $\mathbf{u}$, $\mathbf{v}$ and each of the vectors that you found in part (a), starting each vector at the origin.

Additional Exercise A28

Let $\mathbf{u} = (2, 6)$ and $\mathbf{v} = (4, 2)$.

(a) Determine numbers α and β such that

$$(3, 4) = \alpha\mathbf{u} + \beta\mathbf{v}.$$

(b) Sketch the position vectors $\mathbf{u}$, $\mathbf{v}$ and $(3, 4)$ on a single diagram.

Additional Exercise A29

(a) Determine the vector form of the equation of the line l through the points $(2, 3)$ and $(5, -1)$.

(b) Hence determine whether the points $(7, 2)$ and $(-1, 7)$ lie on l.

Additional Exercise A30

(a) Determine the vector form of the equation of the line l through the points $(1, 1, 1)$ and $(-1, -2, 3)$.

(b) Hence determine whether the points $(5, 7, -3)$ and $(0, -1, 4)$ lie on l.

Additional Exercise A31

Find the angle between the vectors in each of the following pairs.

(a) $(3, 1)$, $(1, -2)$ (b) $\mathbf{i} + 3\mathbf{j}$, $\mathbf{i} - \mathbf{j}$

(c) $(1, 1, 1)$, $(1, 2, 1)$ (d) $(1, 1, 1)$, $(-1, -2, -1)$

(e) $\mathbf{i} + 2\mathbf{j}$, $-3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Additional Exercise A32 Challenging

Prove the multiples property in Subsection 4.4 of Unit A1 in the case where α is negative. That is, prove that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and α is a negative real number, then

$$(\alpha\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\alpha\mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v}).$$

Hint: The following trigonometric identity $\cos(\pi - \theta) = -\cos\theta$, should be useful.

Additional Exercise A33 Challenging

Determine the two unit vectors that make an angle of $\pi/4$ with the vector $\mathbf{p} = (2, 1)$. Verify that the two vectors that you have found are perpendicular to each other.

Additional Exercise A34

Determine the equation of the plane that contains the point $(-1, 3, 2)$ and has the vector $(1, 2, -1)$ as a normal.

Solutions to additional exercises for Unit A1

Solution to Additional Exercise A1

(a) Since $(-2,-4)$ and $(1,6)$ lie on the line, its gradient is

$$m=\frac{(-4)-6}{(-2)-1}=\frac{10}{3}.$$

Then, since the point $(1,6)$ lies on the line, its equation must be

$$y-6=\tfrac{10}{3}(x-1),$$

which can be simplified to

$$3y-18=10x-10,$$

that is,

$$10x-3y=-8.$$

This equation is of the desired form, with $a=10$, $b=-3$ and $c=-8$. (Any multiple of these numbers is also a valid answer.)

(b) Since the line passes through the origin and the point $(7,3)$, it has an equation of the form $y=mx$, where m is the gradient. The coordinates of $(7,3)$ must satisfy the equation $y=mx$. Thus $3=7m$, so $m=\frac{3}{7}$.

Hence the equation of the line is

$$y=\tfrac{3}{7}x.$$

This can be written as

$$3x-7y=0,$$

which is of the desired form, with $a=3$, $b=-7$ and $c=0$.

Solution to Additional Exercise A2

The gradients of the lines $3x+4y+7=0$ and $2x+ky=3$ are

$$-\frac{3}{4}\ \text{and}\ -\frac{2}{k},\quad (k\neq 0),$$

respectively. Thus the lines are

(a) parallel if $-\dfrac{3}{4}=-\dfrac{2}{k}$, that is, if $k=\frac{8}{3}$;

(b) perpendicular if $\left(-\dfrac{3}{4}\right)\times\left(-\dfrac{2}{k}\right)=-1$, that is, if $k=-\frac{3}{2}$.

Solution to Additional Exercise A3

The equation of the circle is

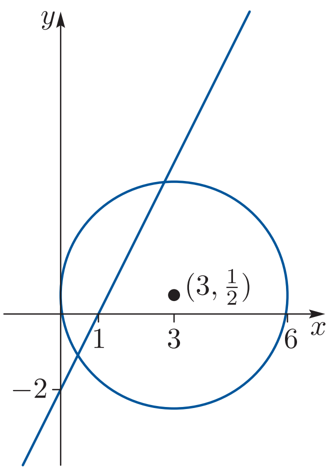
$$(x-2)^2+(y-1)^2=9$$

Solution to Additional Exercise A4

The equation $2x-y-2=0$ is equivalent to $y=2x-2$, which has gradient 2 and y-intercept -2.

The centre of the circle is $(3,\frac{1}{2})$ and the radius is 3.

We use this information to sketch the line and circle.



Solution to Additional Exercise A5

From Additional Exercise A4, we have that the y-intercept of the line is -2 and the centre of the circle is $(3,\frac{1}{2})$. Denote these by A and B respectively.

We use the distance formula for $\mathbb{R}^2$ to get

$$\begin{aligned}AB&=\sqrt{(3-0)^2+\left(\tfrac{1}{2}-(-2)\right)^2}\\&=\sqrt{9+\left(\tfrac{5}{2}\right)^2}\\&=\sqrt{\tfrac{61}{4}}=\tfrac{1}{2}\sqrt{61}.\end{aligned}$$

Solution to Additional Exercise A6

We use the distance formula for $\mathbb{R}^3$ to get

$$\begin{aligned}&\sqrt{(-2-1)^2+(3-(-2))^2+(-1-3)^2}\\&=\sqrt{9+25+16}\\&=\sqrt{50}=5\sqrt{2}.\end{aligned}$$

Solution to Additional Exercise A7

(a) False: 0 is not a natural number.

(b) True: 0 is a rational number.

(c) True: 37 is an integer.

(d) False: -0.6 is a real number.

(e) False: 20 is not a member of the given set.

(f) True: 4, 8 and 16 are all in the given set.

(g) False: $\varnothing$ does not contain any elements.

(h) True: the set $\{1, 2\}$ is the same as the set $\{2, 1\}$.

(i) False: the order pair $(1, 2)$ is not the same as the ordered pair $(2, 1)$.

Solution to Additional Exercise A8

(a) The elements are $3, 4, 5, 6$.

(Note that 2 and 7 are not included.)

(b) The elements are $-1, -4$. These are the solutions of the equation.

(c) The only element is 5. The equation has two solutions, -5 and 5, but only $5 \in \mathbb{N}$.

Solution to Additional Exercise A9

In each case, you may have found a different expression for the set.

(a) $\{k \in \mathbb{Z} : -20 < k < -3\}$

(b) $\{3k : k \in \mathbb{Z},\ k \neq 0\}$

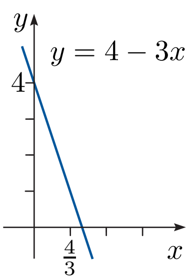
(c) $\{x \in \mathbb{R} : x > 15\}$

(d) $\{x \in \mathbb{R} : -\pi < x \leq 2\sqrt{2}\}$

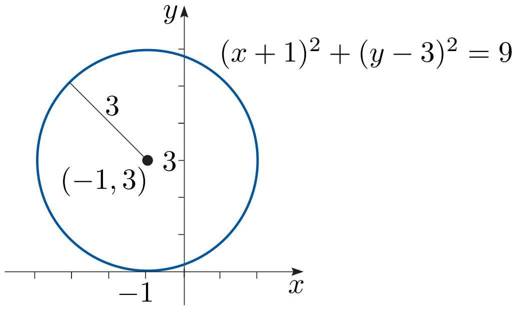
(e) $\{x \in \mathbb{R} : x \geq 71\}$

Solution to Additional Exercise A10

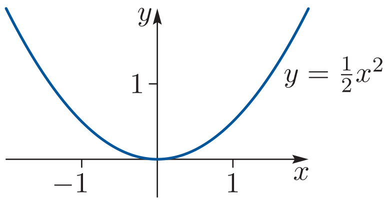
(a) The line $y = 4 - 3x$.



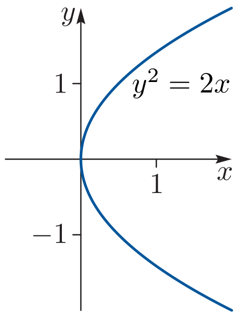
(b) The circle with centre $(-1, 3)$ and radius 3.



(c) The parabola $y = \frac{1}{2}x^2$.

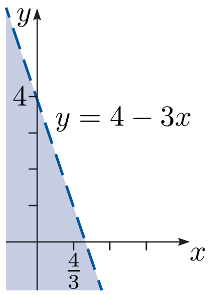


(d) The parabola $y^2 = 2x$.

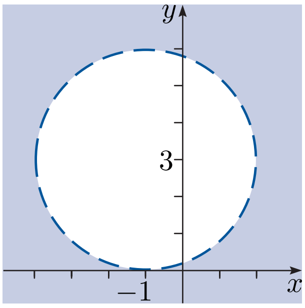


Solution to Additional Exercise A11

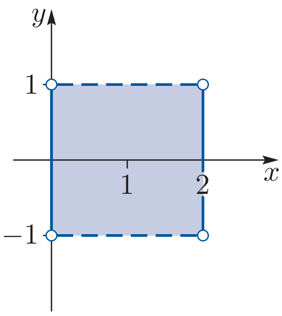
(a)



(b)



(c)



Solution to Additional Exercise A12

(a) $(0,0)$, $(0,6)$ and $(-4,6)$ all satisfy the equation $(x+2)^2+(y-3)^2=13$, so $A \subseteq B$.

(b) The point $(1,0)$ belongs to A but not to B, so A is not a subset of B: $A \nsubseteq B$.

Solution to Additional Exercise A13

We must first show that $A \subseteq B$. Let (x,y) be an arbitrary element of A; then $(x,y) \in \mathbb{R}^2$ and $x^2+4y^2<1$. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, this implies that $4y^2<1$, and hence $y^2<\frac{1}{4}$. Hence $y<\frac{1}{2}$. Thus $(x,y) \in B$.

To confirm that A is a *proper* subset of B, we must show that there is an element of B that does not lie in A. The point $(1,-1)$, for example, lies in B, since $-1<\frac{1}{2}$, but does not lie in A, since

$$1^2+4(-1)^2=5,$$

which is not less than 1. Therefore A is a proper subset of B: $A \subset B$.

Solution to Additional Exercise A14

(a) 1, -1, 2 are the three solutions of $x^3-2x^2-x+2=0$, so $A=B$.

(b) First we show that $A \subseteq B$.

Let $(x,y) \in A$; then $(x,y) \in \mathbb{R}^2$, and for some $t \in [0,2\pi]$ we have $x=\cos t$ and $y=2\sin t$. Hence

$$\begin{aligned}4x^2+y^2 &= 4(\cos^2 t)+(2\sin t)^2\\ &= 4\cos^2 t+4\sin^2 t\\ &= 4(\cos^2 t+\sin^2 t)\\ &= 4,\end{aligned}$$

so $(x,y) \in B$ and $A \subseteq B$.

We now show $B \subseteq A$.

Let $(x,y) \in B$; then $4x^2+y^2=4$. We must show that there is an angle t in $[0,2\pi]$ such that $x=\cos t$ and $y=2\sin t$.

Now, dividing $4x^2+y^2=4$ through by 4 gives

$$x^2+\left(\frac{y}{2}\right)^2=1,$$

and substituting $y'=y/2$ gives

$$x^2+(y')^2=1.$$

This means the point (x,y') lies on the unit circle; that is, the point $(x,y/2)$ lies on the unit circle.

If we take t to be the (anticlockwise) angle from the (positive) x-axis to the line joining the point $(x,y/2)$ with the origin, then $t \in [0,2\pi]$ and $x=\cos t$ and $y/2=\sin t$. Hence $x=\cos t$ and $y=2\sin t$, so $(x,y) \in A$ and $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, it follows that $A=B$.

(c) The set B contains some negative numbers (for example, -1) which cannot be expressed as $\dfrac{p}{q}$ for $p,q \in \mathbb{N}$. Hence $A \neq B$. (Also, $0 \in \mathbb{Q}$, but $0 \notin A$.)

Solution to Additional Exercise A15

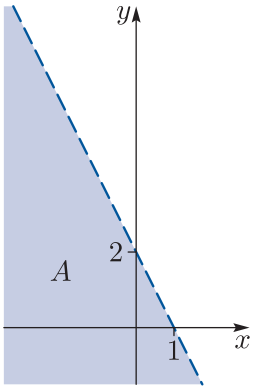
(a) $A \cup B=\{0,2,4,5,6\}$,
$A \cap B=\{4\}$,
$A-B=\{0,2\}$.

(b) $A \cup B=(-5,17]$,
$A \cap B=[2,3]$,
$A-B=(-5,2)$.

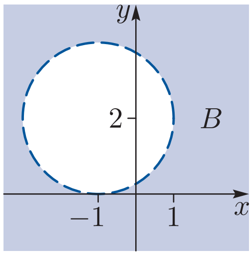
(c) $A \cup B=B$,
$A \cap B=A$,
$A-B=\varnothing$.

Solution to Additional Exercise A16

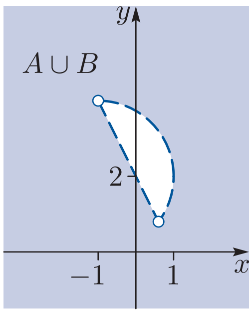
The set A



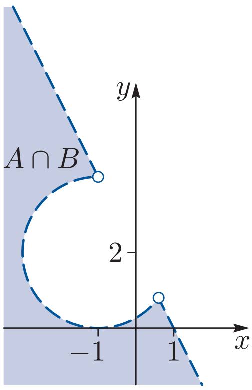
The set B



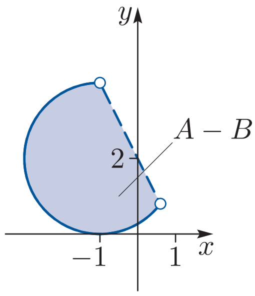
The union $A \cup B$



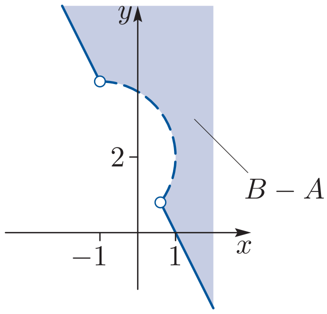
The intersection $A \cap B$



The difference $A - B$

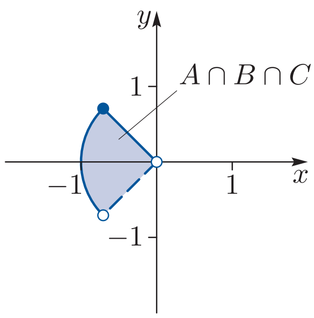


The difference $B - A$

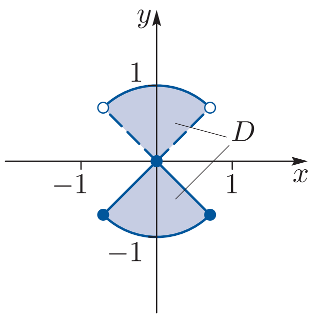


Solution to Additional Exercise A17

The set $A \cap B \cap C$



The set $D = ((A \cap B) \cup (B \cap C)) - (A \cap C)$



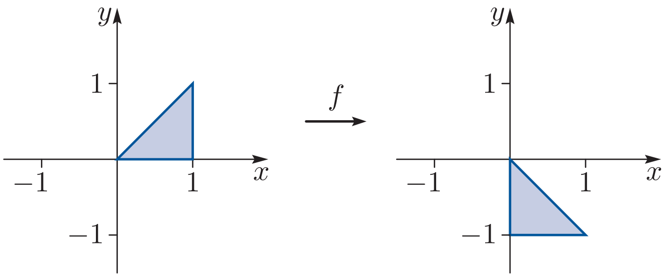
Solution to Additional Exercise A18

(a) This function is the *rotation* of the plane through $3\pi/2$ anticlockwise about the origin.

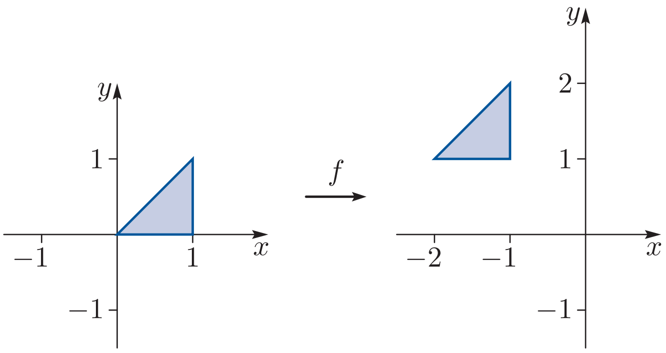
(b) This function is the *translation* of the plane that moves each point to the left by 2 units and up by 1 unit.

Solution to Additional Exercise A19

(a) $f(x, y) = (y, -x)$



(b) $f(x, y) = (x - 2, y + 1)$



Solution to Additional Exercise A20

(a) This function is a rotation so we expect to find that $f(\mathbb{R}^2) = \mathbb{R}^2$.

Let $(x, y) \in \mathbb{R}^2$; then $f(x, y) = (-y, x) \in \mathbb{R}^2$, so $f(\mathbb{R}^2) \subseteq \mathbb{R}^2$.

We must now show that $f(\mathbb{R}^2) \supseteq \mathbb{R}^2$.

Let $(x', y') \in \mathbb{R}^2$. We must show that there exists $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (x', y')$, that is,

$$x' = -y \quad \text{and} \quad y' = x.$$

Rearranging these equations, we obtain

$$x = y' \quad \text{and} \quad y = -x'.$$

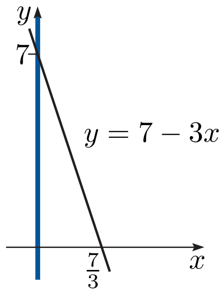
So, for each $(x', y') \in \mathbb{R}^2$, we have

$$(x', y') = f(y', -x'),$$

thus $f(\mathbb{R}^2) \supseteq \mathbb{R}^2$.

Since $f(\mathbb{R}^2) \subseteq \mathbb{R}^2$ and $f(\mathbb{R}^2) \supseteq \mathbb{R}^2$, it follows that $f(\mathbb{R}^2) = \mathbb{R}^2$, so f is onto.

(b) The graph below suggests that $f(\mathbb{R}) = \mathbb{R}$.



We now prove this algebraically.

Let $x \in \mathbb{R}$; then $7 - 3x \in \mathbb{R}$, so $f(\mathbb{R}) \subseteq \mathbb{R}$.

We must now show that $f(\mathbb{R}) \supseteq \mathbb{R}$.

Let $y \in \mathbb{R}$. We must show that there exists $x \in \mathbb{R}$ such that $f(x) = y$; that is, $7 - 3x = y$.

Rearranging gives

$$x = \frac{7 - y}{3},$$

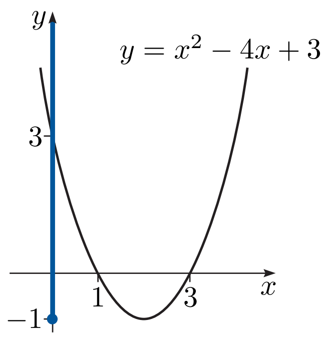
which is in $\mathbb{R}$, so for each $y \in \mathbb{R}$ we have

$$y = f\left(\frac{7 - y}{3}\right).$$

So $f(\mathbb{R}) \supseteq \mathbb{R}$.

Since $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(\mathbb{R}) \supseteq \mathbb{R}$, it follows that $f(\mathbb{R}) = \mathbb{R}$, so f is onto.

(c) The graph below suggests that $f(\mathbb{R}) = [-1, \infty)$.



We now prove this algebraically.

Let $x \in \mathbb{R}$; then

$$\begin{aligned} f(x) &= x^2 - 4x + 3 \\ &= (x - 2)^2 - 1 \geq -1. \end{aligned}$$

So $f(\mathbb{R}) \subseteq [-1, \infty)$.

We must now show that $f(\mathbb{R}) \supseteq [-1, \infty)$.

Let $y \in [-1, \infty)$. We must show that there exists $x \in \mathbb{R}$ such that $f(x) = y$, that is,

$$x^2 - 4x + 3 = y.$$

Rearranging gives

$$(x-2)^2 = y + 1,$$

and we can take $x = 2 + \sqrt{y+1}$, which satisfies $f(x) = y$ and is in $\mathbb{R}$ since $y + 1 \geq 0$.

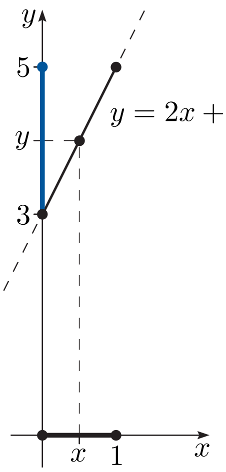
So, for each $y \in [-1, \infty)$, we have $y = f(2 + \sqrt{y+1})$.

Hence $f(\mathbb{R}) \supseteq [-1, \infty)$.

Since $f(\mathbb{R}) \subseteq [-1, \infty)$ and $f(\mathbb{R}) \supseteq [-1, \infty)$, it follows that $f(\mathbb{R}) = [-1, \infty)$.

Since $f(\mathbb{R}) \neq \mathbb{R}$, f is not onto.

(d) The graph below suggests that $f([0,1]) = [3,5]$.



We now prove this algebraically.

Let $x \in [0,1]$. Then $0 \leq x \leq 1$, so $0 \leq 2x \leq 2$, so $3 \leq 2x + 3 \leq 5$. Hence $f(x) \in [3,5]$. Thus $f([0,1]) \subseteq [3,5]$.

We must now show that $f([0,1]) \supseteq [3,5]$.

Let $y \in [3,5]$. We must show that there exists $x \in [0,1]$ such that $f(x) = y$; that is $2x + 3 = y$.

Rearranging gives

$$x = \frac{y-3}{2}.$$

Now $3 \leq y \leq 5$, so $0 \leq y - 3 \leq 2$, so

$$0 \leq \frac{y-3}{2} \leq 1.$$

Thus $(y-3)/2 \in [0,1]$, as required. So for each $y \in [3,5]$ we have

$$y = f\left(\frac{y-3}{2}\right),$$

where $(y-3)/2 \in [0,1]$. So $f([0,1]) \supseteq [3,5]$.

Since $f([0,1]) \subseteq [3,5]$ and $f([0,1]) \supseteq [3,5]$, it follows that $f([0,1]) = [3,5]$. So $f([0,1]) \neq \mathbb{R}$ and f is not onto.

Solution to Additional Exercise A21

(a) This function f is a rotation of the plane, so we expect f to be one-to-one. We now prove this algebraically.

Suppose that $f(x_1, y_1) = f(x_2, y_2)$; then

$$(-y_1, x_1) = (-y_2, x_2),$$

so

$$-y_1 = -y_2 \quad \text{and} \quad x_1 = x_2.$$

Thus $(x_1, y_1) = (x_2, y_2)$, so f is one-to-one.

(b) The graph in the solution to Additional Exercise A20(b) suggests that f is one-to-one. We prove this algebraically.

Suppose that $f(x_1) = f(x_2)$; then

$$7 - 3x_1 = 7 - 3x_2.$$

Thus $x_1 = x_2$, so f is one-to-one.

(c) The graph in the solution to Additional Exercise A20(c) suggests that f is not one-to-one. To show that this is so, we find two points in the domain of f with the same image. For example,

$$f(0) = f(4) = 3,$$

so f is not one-to-one.

(d) The graph in the solution to Additional Exercise A20(d) suggests that f is one-to-one. We prove this algebraically.

Suppose that $f(x_1) = f(x_2)$; then

$$2x_1 + 3 = 2x_2 + 3.$$

Thus $x_1 = x_2$, so f is one-to-one.

Solution to Additional Exercise A22

(a) We have shown in Additional Exercise A21(a) that f is one-to-one, so f has an inverse, and we have shown in the solution to Additional Exercise A20(a) that

$$(x', y') = f(y', -x'),$$

so the inverse of f is the function

$$\begin{aligned} f^{-1} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x', y') &\longmapsto (y', -x'). \end{aligned}$$

This can be expressed in terms of x and y as

$$\begin{aligned} f^{-1} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (y, -x). \end{aligned}$$

(b) We have shown in the solutions to Additional Exercises A20(b) and A21(b) that f is one-to-one and that

$$y = f\left(\frac{7-y}{3}\right), \quad \text{for } y \in \mathbb{R}.$$

Hence f has an inverse

$$\begin{aligned} f^{-1} : \mathbb{R} &\longrightarrow \mathbb{R} \\ y &\longmapsto \frac{7-y}{3}. \end{aligned}$$

This can be expressed in terms of x as

$$\begin{aligned} f^{-1} : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{7-x}{3}. \end{aligned}$$

(c) We have shown in Additional Exercise A21(c) that f is not one-to-one, so f does not have an inverse.

(d) We have shown in the solutions to Additional Exercises A20(d) and A21(d) that f is one-to-one and that the image set of f is $[3, 5]$. We also showed that

$$y = f\left(\frac{y-3}{2}\right), \quad \text{for } y \in [3, 5].$$

Hence f has an inverse

$$\begin{aligned} f^{-1} : [3, 5] &\longrightarrow [0, 1] \\ y &\longmapsto \frac{y-3}{2}. \end{aligned}$$

This can be expressed in terms of x as

$$\begin{aligned} f^{-1} : [3, 5] &\longrightarrow [0, 1] \\ x &\longmapsto \frac{x-3}{2}. \end{aligned}$$

Solution to Additional Exercise A23

(a) Since any number in the domain of g has an image under g which is in $\mathbb{R}$, and hence in the domain of f, the domain of $f \circ g$ is the domain of g.

Also,

$$(f \circ g)(x) = f\left(\frac{1}{x^2 - 4}\right) = 7 - 3\left(\frac{1}{x^2 - 4}\right).$$

Hence the composite is the function

$$\begin{aligned} f \circ g : \mathbb{R} - \{2, -2\} &\longrightarrow \mathbb{R} \\ x &\longmapsto 7 - \frac{3}{x^2 - 4}. \end{aligned}$$

(b) Since any point in the domain of g has an image under g which is in $\mathbb{R}^2$, and hence in the domain of f, the domain of $f \circ g$ is the domain of g.

Also,

$$(f \circ g)(x, y) = f(y, x) = (-x, y).$$

Hence the composite is the function

$$\begin{aligned} f \circ g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-x, y). \end{aligned}$$

Solution to Additional Exercise A24

(a) We note that both functions f and g are one-to-one and onto. Therefore both have inverses. The domain of g is the image set of f, so $g \circ f$ has domain equal to the domain of f.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(x^2) = 9 - x^2, \end{aligned}$$

so

$$\begin{aligned} g \circ f : \{0, 1, 2, 3\} &\longrightarrow \{0, 5, 8, 9\} \\ x &\longmapsto 9 - x^2. \end{aligned}$$

$$f^{-1}(x) = \sqrt{x},$$

so

$$\begin{aligned} f^{-1} : \{0, 1, 4, 9\} &\longrightarrow \{0, 1, 2, 3\} \\ x &\longmapsto \sqrt{x}. \end{aligned}$$

$$g^{-1}(x) = 9 - x,$$

so

$$\begin{aligned} g^{-1} : \{0, 5, 8, 9\} &\longrightarrow \{0, 1, 4, 9\} \\ x &\longmapsto 9 - x. \end{aligned}$$

$$\begin{aligned} (f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x)) \\ &= f^{-1}(9 - x) \\ &= \sqrt{9 - x}, \end{aligned}$$

so

$$\begin{aligned} f^{-1} \circ g^{-1} : \{0,5,8,9\} &\longrightarrow \{0,1,2,3\} \\ x &\longmapsto \sqrt{9-x}. \end{aligned}$$

(b) For each $x \in \{0,5,8,9\}$ we have

$$\begin{aligned} (g \circ f)(f^{-1} \circ g^{-1})(x) &= (g \circ f)(\sqrt{9-x}) \\ &= 9 - (\sqrt{9-x})^2 \\ &= 9 - (9-x) = x, \end{aligned}$$

and for each $x \in \{0,1,2,3\}$ we have

$$\begin{aligned} (f^{-1} \circ g^{-1})(g^\circ f)(x) &= (f^{-1} \circ g^{-1})(9-x^2) \\ &= \sqrt{9-(9-x^2)} \\ &= \sqrt{x^2} = x. \end{aligned}$$

So $(g \circ f)(f^{-1} \circ g^{-1})$ is the identity function on $\{0,5,8,9\}$, and $(f^{-1} \circ g^{-1})(g \circ f)$ is the identity function on $\{0,1,2,3\}$, and hence $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$.

Solution to Additional Exercise A25

Since f and g have inverses, they are both one-to-one and onto (and so are their inverses) and we have

$$f^{-1} : B \longrightarrow A \quad \text{and} \quad g^{-1} : C \longrightarrow B,$$

such that

$$f^{-1} \circ f = i_A \quad \text{and} \quad f \circ f^{-1} = i_B,$$

and

$$g^{-1} \circ g = i_B \quad \text{and} \quad g \circ g^{-1} = i_C.$$

Therefore we have

$$g \circ f : A \longrightarrow C \quad \text{and} \quad f^{-1} \circ g^{-1} : C \longrightarrow A.$$

For any $c \in C$

$$\begin{aligned} (g \circ f)(f^{-1} \circ g^{-1})(c) &= (g \circ f)(f^{-1}(g^{-1}(c))) \\ &= g(f(f^{-1}(g^{-1}(c)))) \\ &= g(g^{-1}(c)) \\ &= c, \end{aligned}$$

since $g^{-1}(c) \in B$ so

$$\begin{aligned} f(f^{-1}(g^{-1}(c)) &= (f \circ f^{-1})(g^{-1}(c)) \\ &= (g^{-1}(c)) \end{aligned}$$

and $c \in C$ so

$$g(g^{-1}(c)) = (g \circ g^{-1})(c) = c.$$

Likewise, for any $a \in A$,

$$\begin{aligned} (f^{-1} \circ g^{-1})(g \circ f)(a) &= (f^{-1} \circ g^{-1}))(g(f(a))) \\ &= f^{-1}(g^{-1}(g(f(a)))) \\ &= f^{-1}(f(a)) \\ &= a, \end{aligned}$$

since $f(a) \in B$ so

$$\begin{aligned} g^{-1}(g(f(a))) &= (g^{-1} \circ g)f(a) \\ &= f(a) \end{aligned}$$

and $a \in A$, so

$$f^{-1}(f(a)) = (f^{-1} \circ f)(a) = a.$$

Since

$$(g \circ f)(f^{-1} \circ g^{-1}) = i_C$$

and

$$(f^{-1} \circ g^{-1})(g \circ f) = i_A,$$

it follows that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$.

Solution to Additional Exercise A26

Since $\mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, we have

$$\begin{aligned} \mathbf{p} + \mathbf{q} &= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k}, \\ \mathbf{p} - \mathbf{q} &= 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} \end{aligned}$$

and

$$\begin{aligned} 2\mathbf{p} - 3\mathbf{q} &= (4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) - (-3\mathbf{i} - 6\mathbf{j} - 12\mathbf{k}) \\ &= 7\mathbf{i} + 14\mathbf{k}. \end{aligned}$$

Solution to Additional Exercise A27

(a) Here,

$$\begin{aligned} \mathbf{u} + 2\mathbf{v} &= (1,1) + 2(2,1) \\ &= (1,1) + (4,2) \\ &= (5,3), \end{aligned}$$

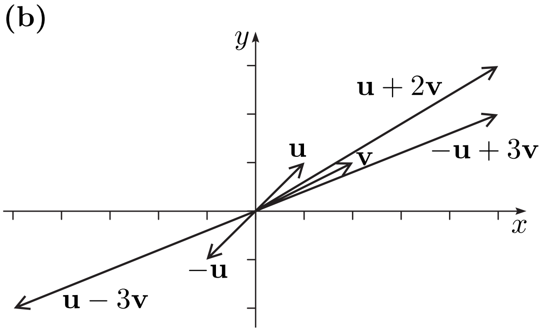
$$-\mathbf{u} = (-1,-1),$$

$$\begin{aligned} -\mathbf{u} + 3\mathbf{v} &= (-1,-1) + 3(2,1) \\ &= (-1,-1) + (6,3) \\ &= (5,2) \end{aligned}$$

and

$$\begin{aligned} \mathbf{u} - 3\mathbf{v} &= (1,1) + (-3)(2,1) \\ &= (1,1) + (-6,-3) \\ &= (-5,-2). \end{aligned}$$

(b)



Solution to Additional Exercise A28

(a) First we write

$$\begin{aligned}\alpha\mathbf{u} + \beta\mathbf{v} &= \alpha(2,6) + \beta(4,2)\\ &= (2\alpha, 6\alpha) + (4\beta, 2\beta)\\ &= (2\alpha + 4\beta, 6\alpha + 2\beta).\end{aligned}$$

Thus

$$(3,4) = (2\alpha + 4\beta, 6\alpha + 2\beta).$$

Then, we equate the first and second components of these vectors; this gives a system of linear equations:

$$\begin{aligned}2\alpha + 4\beta &= 3,\\ 6\alpha + 2\beta &= 4.\end{aligned}$$

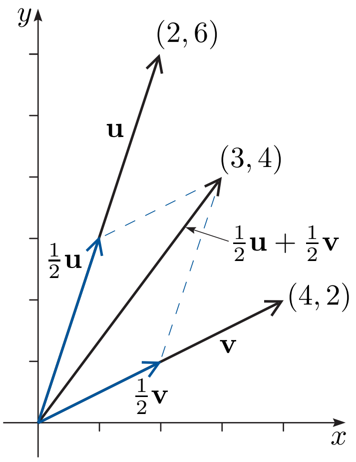
These two equations can be solved to give

$$\alpha = \tfrac{1}{2} \quad \text{and} \quad \beta = \tfrac{1}{2}.$$

Thus we can write

$$(3,4) = \tfrac{1}{2}\mathbf{u} + \tfrac{1}{2}\mathbf{v}.$$

(b)



Solution to Additional Exercise A29

(a) Let $\mathbf{p} = (2,3)$ and $\mathbf{q} = (5,-1)$. Then the vector form of the equation of the line l is

$$\begin{aligned}\mathbf{r} &= (1-\lambda)\mathbf{p} + \lambda\mathbf{q}\\ &= (1-\lambda)(2,3) + \lambda(5,-1)\\ &= (2 - 2\lambda + 5\lambda,\ 3 - 3\lambda - \lambda),\end{aligned}$$

that is,

$$\mathbf{r} = (2 + 3\lambda, 3 - 4\lambda),$$

where $\lambda \in \mathbb{R}$.

(b) The point $(7,2)$ lies on the line l if there is a value of λ such that

$$(7,2) = (2 + 3\lambda, 3 - 4\lambda).$$

Equating components, we obtain

$$7 = 2 + 3\lambda \quad \text{and} \quad 2 = 3 - 4\lambda.$$

The second equation gives $\lambda = \frac{1}{4}$, but this value of λ does not satisfy the first equation. It follows that $(7,2)$ does not lie on the line l.

The point $(-1,7)$ lies on the line l if there is a value of λ such that

$$(-1,7) = (2 + 3\lambda, 3 - 4\lambda).$$

Equating components, we obtain

$$-1 = 2 + 3\lambda \quad \text{and} \quad 7 = 3 - 4\lambda.$$

The second equation gives $\lambda = -1$, and this value also satisfies the first equation. It follows that the point $(-1,7)$ lies on the line l.

Solution to Additional Exercise A30

(a) Let $\mathbf{p} = (1,1,1)$ and $\mathbf{q} = (-1,-2,3)$. Then the vector form of the equation of the line l is

$$\begin{aligned}\mathbf{r} &= (1-\lambda)\mathbf{p} + \lambda\mathbf{q}\\ &= (1-\lambda)(1,1,1) + \lambda(-1,-2,3)\\ &= (1 - \lambda - \lambda,\ 1 - \lambda - 2\lambda,\ 1 - \lambda + 3\lambda),\end{aligned}$$

that is,

$$\mathbf{r} = (1 - 2\lambda,\ 1 - 3\lambda,\ 1 + 2\lambda),$$

where $\lambda \in \mathbb{R}$.

(b) The point $(5,7,-3)$ lies on the line l if there is a value of λ such that

$$(5,7,-3) = (1 - 2\lambda,\ 1 - 3\lambda,\ 1 + 2\lambda).$$

Equating components in turn, we obtain

$$\begin{aligned} 5 &= 1 - 2\lambda, \\ 7 &= 1 - 3\lambda, \\ -3 &= 1 + 2\lambda. \end{aligned}$$

The first equation gives $\lambda = -2$, and this value also satisfies the other two equations. It follows that the point $(5, 7, -3)$ lies on the line l.

The point $(0, -1, 4)$ lies on the line l if there is a value of λ such that

$$(0, -1, 4) = (1 - 2\lambda,\ 1 - 3\lambda,\ 1 + 2\lambda).$$

Equating components in turn, we obtain

$$\begin{aligned} 0 &= 1 - 2\lambda, \\ -1 &= 1 - 3\lambda, \\ 4 &= 1 + 2\lambda. \end{aligned}$$

The first equation gives $\lambda = \frac{1}{2}$, but this value of λ does not satisfy the second equation. It follows that $(0, -1, 4)$ does not lie on the line l.

Solution to Additional Exercise A31

Denote by $\mathbf{p}$ the first vector of each pair, by $\mathbf{q}$ the second vector and by θ the angle between the vectors.

(a) When $\mathbf{p} = (3, 1)$ and $\mathbf{q} = (1, -2)$, we have

$$\begin{aligned} |\mathbf{p}| &= \sqrt{3^2 + 1^2} = \sqrt{10}, \\ |\mathbf{q}| &= \sqrt{1^2 + (-2)^2} = \sqrt{5}, \\ \mathbf{p} \cdot \mathbf{q} &= 3 \times 1 + 1 \times (-2) = 1. \end{aligned}$$

It follows that

$$\begin{aligned} \cos\theta &= \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} \\ &= \frac{1}{\sqrt{50}} \\ &= \frac{1}{5\sqrt{2}}, \end{aligned}$$

so

$$\theta = \cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) = 1.43 \text{ radians} \quad \text{(to 2 d.p.)}.$$

(b) When $\mathbf{p} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{q} = \mathbf{i} - \mathbf{j}$, we have,

$$\begin{aligned} |\mathbf{p}| &= \sqrt{1^2 + 3^2} = \sqrt{10}, \\ |\mathbf{q}| &= \sqrt{1^2 + (-1)^2} = \sqrt{2}, \\ \mathbf{p} \cdot \mathbf{q} &= 1 \times 1 + 3 \times (-1) = -2. \end{aligned}$$

It follows that

$$\begin{aligned} \cos\theta &= \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} \\ &= \frac{-2}{\sqrt{20}} \\ &= \frac{-1}{\sqrt{5}}, \end{aligned}$$

so

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{5}}\right) = 2.03 \text{ radians} \quad \text{(to 2 d.p.)}.$$

(c) When $\mathbf{p} = (1, 1, 1)$ and $\mathbf{q} = (1, 2, 1)$, we have

$$\begin{aligned} |\mathbf{p}| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}, \\ |\mathbf{q}| &= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}, \\ \mathbf{p} \cdot \mathbf{q} &= 1 \times 1 + 1 \times 2 + 1 \times 1 = 4. \end{aligned}$$

It follows that

$$\begin{aligned} \cos\theta &= \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} \\ &= \frac{4}{\sqrt{18}} \\ &= \frac{2\sqrt{2}}{3}, \end{aligned}$$

so

$$\theta = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) = 0.34 \text{ radians} \quad \text{(to 2 d.p.)}.$$

(d) Notice $(-1, -2, -1)$ is $-(1, 2, 1)$ so we use the result from part (c) to get

$$\theta = \pi - 0.34 = 2.80 \text{ radians} \quad \text{(to 2 d.p.)}.$$

(e) When $\mathbf{p} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{q} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, we have

$$\begin{aligned} |\mathbf{p}| &= \sqrt{1^2 + 2^2} = \sqrt{5}, \\ |\mathbf{q}| &= \sqrt{(-3)^2 + 1^2 + (-2)^2} = \sqrt{14}, \\ \mathbf{p} \cdot \mathbf{q} &= 1 \times (-3) + 2 \times 1 + 0 \times (-2) = -1. \end{aligned}$$

It follows that

$$\begin{aligned} \cos\theta &= \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} \\ &= \frac{-1}{\sqrt{70}}, \end{aligned}$$

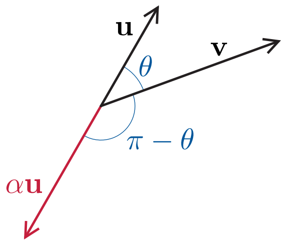
so

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{70}}\right) = 1.69 \text{ radians} \quad \text{(to 2 d.p.)}.$$

Solution to Additional Exercise A32

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and suppose that α is a *negative* constant.

If the angle between $\mathbf{u}$ and $\mathbf{v}$ is θ, then the angle between $\alpha\mathbf{u}$ and $\mathbf{v}$ is $\pi - \theta$, as illustrated below.



So

$$\begin{aligned}(\alpha\mathbf{u}) \cdot \mathbf{v} &= |\alpha\mathbf{u}||\mathbf{v}|\cos(\pi-\theta)\\ &= |\alpha||\mathbf{u}||\mathbf{v}|\cos(\pi-\theta)\\ &= -\alpha|\mathbf{u}||\mathbf{v}|(-\cos\theta)\\ &\qquad (\text{since } |\alpha| = -\alpha\\ &\qquad \text{and } \cos(\pi-\theta) = -\cos\theta)\\ &= \alpha|\mathbf{u}||\mathbf{v}|\cos\theta\\ &= \alpha(\mathbf{u}\cdot\mathbf{v}),\end{aligned}$$

and, similarly,

$$\begin{aligned}\mathbf{u} \cdot (\alpha\mathbf{v}) &= |\mathbf{u}||\alpha\mathbf{v}|\cos(\pi-\theta)\\ &= |\mathbf{u}||\alpha||\mathbf{v}|\cos(\pi-\theta)\\ &= -\alpha|\mathbf{u}||\mathbf{v}|(-\cos\theta)\\ &\qquad (\text{since } |\alpha| = -\alpha\\ &\qquad \text{and } \cos(\pi-\theta) = -\cos\theta)\\ &= \alpha|\mathbf{u}||\mathbf{v}|\cos\theta\\ &= \alpha(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

Solution to Additional Exercise A33

Let such a vector be $\mathbf{r} = (x, y)$. Since $\mathbf{r}$ has magnitude 1 and makes an angle $\pi/4$ with $\mathbf{p} = (2, 1)$, we have

$$\begin{aligned}\mathbf{p}\cdot\mathbf{r} &= |\mathbf{p}||\mathbf{r}|\cos\pi/4\\ &= \sqrt{5}\times 1\times\frac{1}{\sqrt{2}} = \sqrt{\frac{5}{2}}.\end{aligned}$$

In component form we have

$$\mathbf{p}\cdot\mathbf{r} = 2x + y,$$

so

$$2x + y = \sqrt{\tfrac{5}{2}},$$

which gives

$$y = \sqrt{\tfrac{5}{2}} - 2x.$$

Also, $\mathbf{r}$ has magnitude 1, so

$$x^2 + y^2 = 1.$$

Substituting the expression for y into this equation,we obtain

$$x^2 + \left(\sqrt{\tfrac{5}{2}} - 2x\right)^2 = 1,$$

which gives

$$x^2 + \tfrac{5}{2} - 4x\sqrt{\tfrac{5}{2}} + 4x^2 = 1,$$

that is,

$$5x^2 - 2x\sqrt{10} + \tfrac{3}{2} = 0.$$

Multiplying through by 2 gives

$$10x^2 - 4x\sqrt{10} + 3 = 0.$$

By the quadratic formula, this equation has solutions

$$\begin{aligned}x &= \frac{4\sqrt{10} \pm \sqrt{160 - 4\times 10\times 3}}{20}\\ &= \frac{4\sqrt{10} \pm \sqrt{40}}{20}\\ &= \frac{4\sqrt{10} \pm 2\sqrt{10}}{20}\\ &= \frac{2\sqrt{10} \pm \sqrt{10}}{10}.\end{aligned}$$

So the solutions are

$$\frac{3\sqrt{10}}{10} = \frac{3}{\sqrt{10}} \quad\text{and}\quad \frac{\sqrt{10}}{10} = \frac{1}{\sqrt{10}}.$$

The corresponding values of y are, respectively,

$$\begin{aligned}y &= \sqrt{\frac{5}{2}} - 2\times\frac{3}{\sqrt{10}},\\ &= \frac{5}{\sqrt{10}} - \frac{6}{\sqrt{10}} = -\frac{1}{\sqrt{10}},\end{aligned}$$

and

$$\begin{aligned}y &= \sqrt{\frac{5}{2}} - 2\times\frac{1}{\sqrt{10}},\\ &= \frac{5}{\sqrt{10}} - \frac{2}{\sqrt{10}} = \frac{3}{\sqrt{10}}.\end{aligned}$$

So the two unit vectors making an angle of $\pi/4$ with $(2,1)$ are

$$\left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right).$$

The scalar product of these two vectors is

$$\begin{aligned}&\left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right) \cdot \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)\\ &= \frac{3}{10} - \frac{3}{10} = 0,\end{aligned}$$

so the two vectors are perpendicular.

Solution to Additional Exercise A34

The equation of the plane is given by $\mathbf{x} \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n}$, where $\mathbf{x} = (x, y, z)$, $\mathbf{p} = (-1, 3, 2)$ and $\mathbf{n} = (1, 2, -1)$.

So the equation of the plane is

$$(x, y, z) \cdot (1, 2, -1) = (-1, 3, 2) \cdot (1, 2, -1),$$

that is,

$$x + 2y - z = (-1) \times 1 + 3 \times 2 + 2 \times (-1),$$

which simplifies to

$$x + 2y - z = 3.$$

Additional exercises for Unit A2

Section 1

Additional Exercise A35

(a) Show that $\mathbb{N}$ does not satisfy the multiplicative inverses property (M4) by giving an example of a natural number that does not have a multiplicative inverse.

(b) Which natural numbers have a multiplicative inverse in $\mathbb{N}$?

(c) Which of the additive properties A1–A5 do not hold for $\mathbb{N}$?

Additional Exercise A36

(a) Show that $2x^3 + x^2 - 13x + 6$ has a factor $x - 2$, and hence factorise this polynomial.

(b) Solve the equation $x^3 + 6x^2 + 3x - 10 = 0$.

Additional Exercise A37

(a) Find a cubic polynomial for which the sum of the roots is 0, the product of the roots is -30, and one root is 3.

(b) Find the other two roots of the polynomial found in part (a).

Section 2

Additional Exercise A38

Let $z_1 = 2 + 3i$ and $z_2 = 1 - 4i$. Find

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 z_2$ (d) $\overline{z_1}$

(e) $\overline{z_2}$ (f) z_1/z_2 (g) $1/z_1$

Additional Exercise A39

Draw a diagram showing each of the following complex numbers in the complex plane, and express them in polar form, using principal arguments.

(a) $\sqrt{3} - i$ (b) $-5i$ (c) $-2(1 + i\sqrt{3})$

Additional Exercise A40

Express each of the following complex numbers in Cartesian form.

(a) $2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(b) $3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ (c) $\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

Additional Exercise A41

Let $z_1 = \sqrt{3} - i$, $z_2 = -5i$ and $z_3 = -2(1 + i\sqrt{3})$. Use the solution to Additional Exercise A39 to determine the following complex numbers in polar form in terms of the principal argument.

(a) $z_1 z_2 z_3$ (b) $\dfrac{z_1 z_2}{z_3}$

Additional Exercise A42

Solve the equation $z^5 = -32$, leaving your answers in polar form.

Additional Exercise A43

Solve the equation $z^3 + z^2 - z + 15 = 0$, given that one solution is an integer.

Additional Exercise A44

Determine a polynomial of degree 4 whose roots are 3, -2, $2 - i$ and $2 + i$.

Additional Exercise A45

Use the definition of e^z to express the following complex numbers in Cartesian form.

(a) $e^{i\pi/2}$ (b) $e^{3+i\pi/4}$ (c) $e^{-1+i\pi}$

Additional Exercise A46 Challenging

Let $p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$ be a polynomial in z with *real* coefficients, and let α be a root of $p(z)$. Show that the complex conjugate $\overline{\alpha}$ is also a root of $p(z)$.

Section 3

Additional Exercise A47

Evaluate the following sums and products in modular arithmetic.

(a) $21 +_{26} 15$ (b) $21 \times_{26} 15$ (c) $19 +_{33} 14$

(d) $19 \times_{33} 14$

Additional Exercise A48

Use Euclid's Algorithm to find:

(a) the multiplicative inverse of 8 in $\mathbb{Z}_{21}$

(b) the multiplicative inverse of 19 in $\mathbb{Z}_{33}$.

Additional Exercise A49

Construct the multiplication table for $\mathbb{Z}_{11}$, and hence find the multiplicative inverse of every non-zero element in $\mathbb{Z}_{11}$.

Additional Exercise A50

Solve the following equations.

(a) $8 \times_{21} x = 13$ (b) $19 \times_{33} x = 15$

In Additional Exercise A48, you found that $8^{-1} = 8$ in $\mathbb{Z}_{21}$, and $19^{-1} = 7$ in $\mathbb{Z}_{33}$.

Additional Exercise A51

Find all the solutions of the following equations in $\mathbb{Z}_8$.

(a) $3 \times_8 x = 7$ (b) $4 \times_8 x = 7$

(c) $4 \times_8 x = 4$

Additional Exercise A52

Find all the solutions of the following equations in $\mathbb{Z}_{15}$.

(a) $3 \times_{15} x = 6$ (b) $4 \times_{15} x = 3$

(c) $5 \times_{15} x = 2$

Solutions to additional exercises for Unit A2

Solution to Additional Exercise A35

(a) There is no natural number 2^{-1} such that $2 \times 2^{-1} = 2^{-1} \times 2 = 1$, for example.

(b) The number 1 is the only number that has a multiplicative inverse in $\mathbb{N}$. (The multiplicative inverse of 1 is 1.)

(c) There is no additive identity in $\mathbb{N}$, since $0 \notin \mathbb{N}$, so the additive identity property (A3) does not hold. The additive inverses property (A4) therefore cannot hold because there is no zero element. (The other properties (A1, A2 and A5) all hold because they hold for real numbers.

Solution to Additional Exercise A36

(a) Putting $x = 2$, we obtain $16 + 4 - 26 + 6 = 0$, so by the Factor Theorem $x - 2$ is a factor. Hence by comparing coefficients

$$\begin{aligned} 2x^3 + x^2 - 13x + 6 &= (x-2)(2x^2 + 5x - 3) \\ &= (x-2)(x+3)(2x-1). \end{aligned}$$

(b) Trying integer values that are factors of 10, we find that $x = 1$ is a root, so $x - 1$ is a factor. Hence by comparing coefficients

$$\begin{aligned} x^3 + 6x^2 + 3x - 10 &= (x-1)(x^2 + 7x + 10) \\ &= (x-1)(x+2)(x+5), \end{aligned}$$

so the solutions of $x^3 + 6x^2 + 3x - 10 = 0$ are $x = 1$, $x = -2$ and $x = -5$.

Solution to Additional Exercise A37

(a) If the sum of the roots is 0 and the product is -30, then the cubic polynomial must be of the form

$$x^3 + cx + 30, \quad \text{for some } c \in \mathbb{R}.$$

If $x = 3$ is a root, then

$$27 + 3c + 30 = 0,$$

so $c = -19$.

Hence the polynomial is $x^3 - 19x + 30$.

(b) We know that $x - 3$ is a factor. Hence

$$\begin{aligned} x^3 - 19x + 30 &= (x-3)(x^2 + 3x - 10) \\ &= (x-3)(x-2)(x+5), \end{aligned}$$

so the other two roots are 2 and -5.

Solution to Additional Exercise A38

(a) $$\begin{aligned} z_1 + z_2 &= (2+3i) + (1-4i) \\ &= 3 - i. \end{aligned}$$

(b) $$\begin{aligned} z_1 - z_2 &= (2+3i) - (1-4i) \\ &= 1 + 7i. \end{aligned}$$

(c) $$\begin{aligned} z_1 z_2 &= (2+3i)(1-4i) \\ &= 2 + 3i - 8i - 12i^2 \\ &= 14 - 5i. \end{aligned}$$

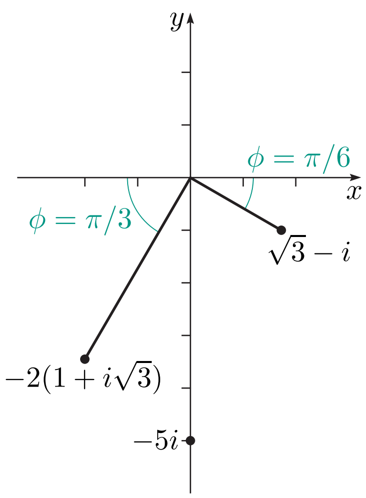
(d) $\overline{z_1} = 2 - 3i$.

(e) $\overline{z_2} = 1 + 4i$.

(f) $$\begin{aligned} \frac{z_1}{z_2} = \frac{2+3i}{1-4i} &= \frac{(2+3i)(1+4i)}{(1-4i)(1+4i)} \\ &= \frac{-10+11i}{1^2 + (-4)^2} \\ &= -\tfrac{1}{17}(10 - 11i). \end{aligned}$$

(g) $$\begin{aligned} \frac{1}{z_1} = \frac{1}{2+3i} &= \frac{2-3i}{(2+3i)(2-3i)} \\ &= \frac{2-3i}{2^2+3^2} \\ &= \tfrac{1}{13}(2-3i). \end{aligned}$$

Solution to Additional Exercise A39



(a) Let $z = x + iy = \sqrt{3} - i$, so $x = \sqrt{3}$ and $y = -1$. Then $z = r(\cos\theta + i\sin\theta)$, where

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2.$$

To find θ, we calculate

$$\cos\phi = \frac{|x|}{r} = \frac{\sqrt{3}}{2}.$$

So $\phi = \pi/6$, and from the diagram we see that $\theta = -\phi = -\pi/6$. Hence the polar form of $\sqrt{3} - i$ in terms of the principal argument is

$$2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).$$

(b) Let $z = x + iy = -5i$, so $x = 0$ and $y = -5$. Then $z = r(\cos\theta + i\sin\theta)$, where

$$r = \sqrt{0^2 + (-5)^2} = 5.$$

Also z lies on the negative half of the imaginary axis, so $\theta = -\pi/2$.

Hence the polar form of $-5i$ in terms of the principal argument is

$$5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right).$$

(c) Let $z = x + iy = -2(1 + i\sqrt{3})$, so $x = -2$ and $y = -2\sqrt{3}$. Then $z = r(\cos\theta + i\sin\theta)$, where

$$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4.$$

To find θ, we calculate

$$\cos\phi = \frac{|x|}{r} = \frac{1}{2}.$$

So $\phi = \pi/3$, and from the diagram we see that

$$\theta = -(\pi - \phi) = -\frac{2\pi}{3}.$$

Hence the polar form of $-2(1 + i\sqrt{3})$ in terms of the principal argument is

$$4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right).$$

Solution to Additional Exercise A40

(a) The required form is $x + iy$, where

$$x = 2\sqrt{2}\cos\frac{\pi}{4} = 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 2$$

and

$$y = 2\sqrt{2}\sin\frac{\pi}{4} = 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 2,$$

so the Cartesian form is $2 + 2i$.

(b) The required form is $x + iy$, where

$$x = 3\cos\frac{\pi}{2} = 0 \quad \text{and} \quad y = 3\sin\frac{\pi}{2} = 3,$$

so the Cartesian form is $3i$.

(c) The required form is $x + iy$, where

$$x = \cos\frac{5\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

and

$$y = \sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2},$$

so the Cartesian form is $\frac{1}{2}\left(-\sqrt{3} + i\right)$.

Solution to Additional Exercise A41

From the solution to Additional Exercise A39, we have

$$z_1 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right),$$

$$z_2 = 5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right),$$

$$z_3 = 4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right).$$

(a) Hence

$$\begin{aligned} z_1z_2z_3 &= 2 \times 5 \times 4\left(\cos\left(-\frac{\pi}{6}\pi - \frac{\pi}{2} - \frac{2\pi}{3}\right)\right. \\ &\qquad \left. + i\sin\left(-\frac{\pi}{6}\pi - \frac{\pi}{2} - \frac{2\pi}{3}\right)\right) \\ &= 40\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right) \\ &= 40\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right), \end{aligned}$$

using the principal argument.

(b) Hence

$$\begin{aligned} \frac{z_1z_2}{z_3} &= \frac{2 \times 5}{4}\left(\cos\left(-\frac{\pi}{6}\pi - \frac{\pi}{2} - \left(-\frac{2\pi}{3}\right)\right)\right. \\ &\qquad \left. + i\sin\left(-\frac{\pi}{6}\pi - \frac{\pi}{2} - \left(-\frac{2\pi}{3}\right)\right)\right) \\ &= \frac{5}{2}(\cos 0 + i\sin 0) = \frac{5}{2}. \end{aligned}$$

Solution to Additional Exercise A42

Let $z = r(\cos\theta + i\sin\theta)$.

Then, since $-32 = 32(\cos\pi + i\sin\pi)$, we have

$$r^5(\cos 5\theta + i\sin 5\theta) = 32(\cos\pi + i\sin\pi).$$

Hence $r = 2$ and $\theta = \dfrac{\pi}{5} + \dfrac{2k\pi}{5}$ for any integer k, and the five solutions of $z^5 = -32$ are given by

$$z = 2\left(\cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)\right)$$

for $k = 0, 1, 2, 3, 4$.

Hence the solutions are

$$\begin{aligned}
z_0 &= 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right),\\
z_1 &= 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right),\\
z_2 &= 2(\cos\pi + i\sin\pi) = -2,\\
z_3 &= 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)\\
&= 2\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right),\\
z_4 &= 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)\\
&= 2\left(\cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)\right).
\end{aligned}$$

Solution to Additional Exercise A43

The integer solution must be a factor of the constant term 15, so it must be one of ±1, ±3, ±5, ±15. (See the end of Subsection 1.4 of Unit A2.)

Testing these, we find $z = -3$ is a root, since

$$(-3)^3 + (-3)^2 - (-3) + 15 = 0.$$

Hence $z + 3$ is a factor, and we find that

$$z^3 + z^2 - z + 15 = (z+3)(z^2 - 2z + 5).$$

The solutions of $z^2 - 2z + 5 = 0$ are given by

$$z = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Hence the solutions of $z^3 + z^2 - z + 15 = 0$ are $z = -3$, $z = 1 + 2i$ and $z = 1 - 2i$.

Solution to Additional Exercise A44

A suitable polynomial is

$$(z-3)(z+2)\left(z-(2-i)\right)\left(z-(2+i)\right),$$

that is,

$$(z^2 - z - 6)(z^2 - 4z + 5)$$

or

$$z^4 - 5z^3 + 3z^2 + 19z - 30.$$

Solution to Additional Exercise A45

(a) $e^{i\pi/2} = e^{0+i\pi/2} = e^0\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$
$= 1(0+i) = i.$

(b) $e^{3+i\pi/4} = e^3\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$
$= \dfrac{e^3}{\sqrt{2}} + i\dfrac{e^3}{\sqrt{2}}.$

(c) $e^{-1+i\pi} = e^{-1}(\cos\pi + i\sin\pi) = -e^{-1}.$

Solution to Additional Exercise A46

Since α is a root, we have $p(\alpha) = 0$; that is,

$$a_n\alpha^n + \cdots + a_2\alpha^2 + a_1\alpha + a_0 = 0.$$

To show $\overline{\alpha}$ is a root, we need to show that $p(\overline{\alpha}) = 0$ also.

Now, using the properties of complex conjugates given in Subsection 2.2 of Unit A2, and the fact that the conjugate of a real number is just itself, we have

$$\begin{aligned}
p(\overline{\alpha}) &= a_n\overline{\alpha}^n + \cdots + a_2\overline{\alpha}^2 + a_1\overline{\alpha} + a_0\\
&= a_n\overline{\alpha^n} + \cdots + a_2\overline{\alpha^2} + a_1\overline{\alpha} + a_0\\
&= \overline{a_n\alpha^n} + \cdots + \overline{a_2\alpha^2} + \overline{a_1\alpha} + \overline{a_0}\\
&= \overline{a_n\alpha^n + \cdots + a_2\alpha^2 + a_1\alpha + a_0}\\
&= \overline{0} = 0.
\end{aligned}$$

So $p(\overline{\alpha}) = 0$ and the complex conjugate $\overline{\alpha}$ is a root of a polynomial with real coefficients whenever α is.

Solution to Additional Exercise A47

(a) $21 +_{26} 15 = 10$

(We have $21 + 15 = 36 = 26 + 10$.)

(b) $21 \times_{26} 15 = 3$

(We have $21 \times 15 = 315 = 260 + 52 + 3$.)

(c) $19 +_{33} 14 = 0$

(We have $19 + 14 = 33 + 0$.)

(d) $19 \times_{33} 14 = 2$

(We have $19 \times 14 = 266 = 99 + 99 + 66 + 2$.)
Alternatively,

$$\begin{aligned}
19 \times 14 &\equiv 19 \times 2 \times 7\\
&\equiv 38 \times 7\\
&\equiv 5 \times 7\\
&\equiv 35 \equiv 2 \pmod{33}.
\end{aligned}$$

Solution to Additional Exercise A48

(a) We have

$$\begin{aligned} 21 &= 2 \times 8 + 5 \\ 8 &= 1 \times 5 + 3 \\ 5 &= 1 \times 3 + 2 \\ 3 &= 1 \times 2 + 1 \end{aligned}$$

Hence

$$\begin{aligned} 1 &= 3 - 1 \times 2 \\ &= 3 - (5 - 3) \\ &= -5 + 2 \times 3 \\ &= -5 + 2 \times (8 - 5) \\ &= 2 \times 8 - 3 \times 5 \\ &= 2 \times 8 - 3 \times (21 - 2 \times 8) \\ &= -3 \times 21 + 8 \times 8. \end{aligned}$$

Hence $8 \times 8 = 3 \times 21 + 1$, so

$$8 \times_{21} 8 = 1,$$

so the multiplicative inverse of 8 in $\mathbb{Z}_{21}$ is 8.

(b) We have

$$\begin{aligned} 33 &= 1 \times 19 + 14 \\ 19 &= 1 \times 14 + 5 \\ 14 &= 2 \times 5 + 4 \\ 5 &= 1 \times 4 + 1 \end{aligned}$$

Hence

$$\begin{aligned} 1 &= 5 - 4 \\ &= 5 - (14 - 2 \times 5) \\ &= -14 + 3 \times 5 \\ &= -14 + 3 \times (19 - 14) \\ &= 3 \times 19 - 4 \times 14 \\ &= 3 \times 19 - 4 \times (33 - 19) \\ &= -4 \times 33 + 7 \times 19. \end{aligned}$$

Hence

$$7 \times 19 = 4 \times 33 + 1,$$

so

$$7 \times_{33} 19 = 1,$$

so the multiplicative inverse of 19 in $\mathbb{Z}_{33}$ is 7.

Solution to Additional Exercise A49

$\times_{11}$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

Hence we have the following multiplicative inverses in $\mathbb{Z}_{11}$.

x	1	2	3	4	5	6	7	8	9	10
x^{-1}	1	6	4	3	9	2	8	7	5	10

Solution to Additional Exercise A50

(a) The given equation is

$$8 \times_{21} x = 13.$$

Multiplying both sides by the multiplicative inverse in $\mathbb{Z}_{21}$, which is 8, gives

$$8 \times_{21} (8 \times_{21} x) = 8 \times_{21} 13,$$

so $x = 8 \times_{21} 13$.

Now, $8 \times 13 = 104 = 5 \times 21 - 1$, and $-1 \equiv 20 \pmod{21}$, so the solution is $x = 20$.

(b) The given equation is

$$19 \times_{33} x = 15.$$

Multiplying both sides by the multiplicative inverse in $\mathbb{Z}_{33}$, which is 7, gives

$$7 \times_{33} (19 \times_{33} x) = 7 \times_{33} 15,$$

so $x = 7 \times_{33} 15$.

Now, $7 \times 15 = 105 = 99 + 6$, so the solution is $x = 6$.

Solution to Additional Exercise A51

(a) The equation $3 \times_8 x = 7$ has a unique solution because 3 and 8 are coprime.

The solution, $x = 5$, can be found in various ways: for example, by noticing that $3 \times 3 = 9 = 8 + 1$, so $3^{-1} = 3$ in $\mathbb{Z}_8$ and calculating $x = 3^{-1} \times_8 7 = 5$, or by spotting that $7 \equiv 15 \pmod{8}$ and therefore $3 \times_8 5 = 7$, or by testing possible values for x.

(b) The equation $4 \times_8 x = 7$ has no solutions, since the highest common factor of 4 and 8 is 4, but this is not a factor of 7.

(c) The equation $4 \times_8 x = 4$ has $d = 4$ solutions, since 4 is the highest common factor of 4 and 8, and it is also a factor of 4.

One solution of this equation is $x = 1$.

Also, $n/d = 8/4 = 2$, so the other solutions are $x = 1 + 2 = 3$, $x = 1 + 2 \times 2 = 5$ and $x = 1 + 3 \times 2 = 7$.

Solution to Additional Exercise A52

(a) The equation $3 \times_{15} x = 6$ has $d = 3$ solutions because 3 is the highest common factor of 3 and 15, and it also a factor of 6.

One solution of this equation is $x = 2$.

Also $n/d = 15/3 = 5$, so the other solutions are $x = 2 + 5 = 7$ and $x = 2 + 2 \times 5 = 12$.

(b) The equation $4 \times_{15} x = 3$ has a unique solution because 4 and 15 are coprime.

The solution, $x = 12$, can be found in various ways: for example, by noticing that $48 = 3 \times 15 + 3$ so $48 \equiv 3 \pmod{15}$ and therefore $4 \times_{15} 12 = 3$, or by noticing that $4 \times 4 = 16 = 15 + 1$, so $4^{-1} = 4$ in $\mathbb{Z}_{15}$ and calculating $x = 4^{-1} \times_{16} 3 = 12$, or by spotting that $3 \equiv -12 \pmod{15}$ and hence $4 \times (-3) \equiv 3 \pmod{15}$, and since $-3 \equiv 12 \pmod{15}$, we have $4 \times_{15} 12 = 3$, or by testing possible values for x.

(c) The equation $5 \times_{15} x = 2$ has no solutions, since the highest common factor of 5 and 15 is 5, but this is not a factor of 2.

Additional exercises for Unit A3

Section 1

Additional Exercise A53

Which of the following statements have the same meaning?

(a) If n is even, then n^2 is a multiple of 4.

(b) n is even only if n^2 is a multiple of 4.

(c) n^2 is a multiple of 4 whenever n is even.

(d) $x > 0 \implies x^2 + 4x > 0$.

(e) $x > 0$ is necessary for $x^2 + 4x > 0$.

(f) $x > 0$ is sufficient for $x^2 + 4x > 0$.

Additional Exercise A54

Consider the following implication:

if x divides 6 or x divides 8, then x divides 48.

Write down

(a) the negation

(b) the converse

(c) the contrapositive.

Additional Exercise A55

Write down the negation of each of the following statements.

(a) $\forall x,\ x \in A \implies x \notin B$.

(b) $\exists x$ such that $x > 0$ and $x < 0$.

Section 2

Additional Exercise A56

Determine whether the numbers 221 and 223 are prime.

(In Exercise A124 of Unit A3 you proved that if an integer $n > 1$ is not divisible by any of the primes less than or equal to $\sqrt{n}$, then n is a prime number.)

Additional Exercise A57

Prove, or give a counterexample to disprove, each of the following statements.

(a) If n is a positive integer, then $n^3 - n$ is even.

(b) If $m + n$ is a multiple of k, then m and n are multiples of k.

(c) If θ is a real number, then $\sin 2\theta = 2\sin\theta$.

(d) The following function is one-to-one:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 3x^2 - 6x + 1. \end{aligned}$$

(e) The function g is the inverse of the function f, where f and g are given by

$$\begin{aligned} f : \mathbb{R} - \{1\} &\longrightarrow \mathbb{R} - \{0\} \\ x &\longmapsto \frac{1}{x-1} \end{aligned} \quad \text{and}$$

$$\begin{aligned} g : \mathbb{R} - \{0\} &\longrightarrow \mathbb{R} - \{1\} \\ x &\longmapsto 1 + \frac{1}{x}. \end{aligned}$$

Additional Exercise A58

(a) Write down the converse of the following statement.

If m and n are both even integers, then $m - n$ is an even integer.

(b) Determine whether the original statement and the converse are true, and give a proof or counterexample, as appropriate.

Additional Exercise A59

Prove each of the following statements by mathematical induction.

(a) $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \cdots + \dfrac{1}{(n-1)n} = \dfrac{n-1}{n}$, for $n = 2, 3, \ldots$.

(b) The integer $3^{2n} - 1$ is divisible by 8, for $n = 1, 2, \ldots$.

Additional Exercise A60

Determine which of the following statements are true, and give a proof or counterexample as appropriate.

(a) For all $x, y \in \mathbb{R}$, $x < y \implies x^2 < y^2$.

(b) For all $x \in \mathbb{R}$, $x^2 - x = 2$.

(c) There exists $x \in \mathbb{R}$ such that $x^2 - x = 2$.

(d) There exists $x \in \mathbb{R}$ such that $x^2 - x = -1$.

(e) There are no real numbers x, y for which x/y and y/x are both integers.

(f) For all positive integers n,
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

(g) For all positive integers $n \geq 2$,
$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{2n}.$$

Section 3

Additional Exercise A61

Prove by contradiction that $(a+b)^2 \geq 4ab$ for all real numbers a and b.

Additional Exercise A62

(a) Write down the contrapositive of the following statement, for positive integers n.

If n^2 is divisible by 3, then n is divisible by 3.

(b) Prove that the contrapositive is true, and hence that the original statement is true.

Section 4

Additional Exercise A63

Let $\sim$ be the relation defined on $\mathbb{Z}$ by
$$m \sim n \quad \text{if } m + n \text{ is even.}$$
Determine whether $\sim$ is reflexive, symmetric and transitive, and state whether it is an equivalence relation.

Additional Exercise A64

Let $\sim$ be the relation defined on $\mathbb{Z}$ by
$$x \sim y \quad \text{if } 2x - y \text{ is divisible by 7.}$$
Show that $\sim$ is not reflexive, symmetric or transitive.

Additional Exercise A65

Let A be the set of all functions with domain and codomain $\mathbb{R}$, and let $\sim$ be the relation defined on A by
$$f \sim g \quad \text{if } f(0) = g(0).$$

(a) Show that $\sim$ is an equivalence relation.

(b) Describe the equivalence classes, and describe a set of representatives for the equivalence classes.

Additional Exercise A66

Let $\sim$ be the relation defined on $\mathbb{C}$ by
$$z_1 \sim z_2 \quad \text{if } x_1 - x_2 = 5(y_1 - y_2),$$
where
$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2.$$
Show that this is an equivalence relation and describe the equivalence classes.

Additional Exercise A67

Let $\sim$ be the relation $\sim$ defined on $\mathbb{N}$ by
$$m \sim n \quad \text{if } \frac{m}{n} = 10^k \text{ for some integer } k.$$
Show that $\sim$ is an equivalence relation.

Additional Exercise A68 Challenging

For each of the following equivalence relations, describe a set of representatives for its equivalence classes, and describe its equivalence classes.

(a) The relation $\sim$ defined on $\mathbb{N}$ by

$$m \sim n \quad \text{if } \frac{m}{n} = 10^k \text{ for some integer } k.$$

(You were asked to show that $\sim$ is an equivalence relation in Additional Exercise A67.)

(b) The relation $\sim$ defined on $\mathbb{R}$ by

$$x \sim y \quad \text{if } x - y \text{ is an integer.}$$

(You saw that $\sim$ is an equivalence relation in Worked Exercise A74(b) in Unit A3, and the equivalence class $[\![3.7]\!]$ was found in Worked Exercise A75, also in Unit A3. The relation $\sim$ is congruence modulo 1 on $\mathbb{R}$.)

Solutions to additional exercises for Unit A3

Solution to Additional Exercise A53

(a), (b) and (c) all have the same meaning.

(d) and (f) have the same meaning.

(You may like to show that (a) is true, and hence that (b) and (c) are true; that (d) and hence (f) are true, but (e) is false.)

Solution to Additional Exercise A54

(a) The negation is

x divides 6 or x divides 8, and x does not divide 48.

(b) The converse is

if x divides 48, then x divides 6 or x divides 8.

(c) The negation of 'x divides 6 or x divides 8' is

x does not divide 6 and x does not divide 8,

so the contrapositive is

if x does not divide 48, then x does not divide 6 and x does not divide 8.

Solution to Additional Exercise A55

(a) The negation of '$x \in A \implies x \notin B$' is

$x \in A$ and $x \in B$,

therefore the negation is

$\exists x$ such that $x \in A$ and $x \in B$.

(b) The negation of '$x > 0$ and $x < 0$' is

$x \leq 0$ or $x \geq 0$,

therefore the negation is

$\forall x$, $x \leq 0$ or $x \geq 0$.

Solution to Additional Exercise A56

We have $13^2 = 169$ and $17^2 = 289$, so we only need to check the primes 2, 3, 5, 7, 11, 13.

221 is divisible by 13 ($221 = 13 \times 17$), so it is not prime.

223 is not divisible by any of 2, 3, 5, 7, 11 and 13, so it is prime.

Solution to Additional Exercise A57

(a) This statement is true.

We have

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1).$$

Either n is even or $n+1$ is even, so $n^3 - n$ is even.

(b) This statement is false.

For example, $6 + 4$ is a multiple of 5, but 6 and 4 are not multiples of 5.

(c) This statement is false.

For example, if $\theta = \pi/2$, then

$$\sin 2\theta = \sin \pi = 0,$$

but

$$2\sin\theta = 2\sin(\pi/2) = 2.$$

(d) This statement is false.

For example, $f(0) = f(2) = 1$.

(e) This statement is true.

We follow Strategy A2 in Unit A1 and show that $f \circ g$ is the identity function on $\mathbb{R} - \{0\}$ and that $g \circ f$ is the identity function on $\mathbb{R} - \{1\}$.

We have

$$(f \circ g)(x) = f\left(1 + \frac{1}{x}\right) = \frac{1}{\left(1 + \frac{1}{x}\right) - 1} = x$$

and

$$f \circ g : \mathbb{R} - \{0\} \longrightarrow \mathbb{R} - \{0\}.$$

Also,

$$\begin{aligned}(g \circ f)(x) = g\left(\frac{1}{x-1}\right) &= 1 + \frac{1}{1/(x-1)} \\ &= 1 + x - 1 = x\end{aligned}$$

and

$$g \circ f : \mathbb{R} - \{1\} \longrightarrow \mathbb{R} - \{1\}.$$

Hence, from Strategy A2, g is the inverse of f.

Solution to Additional Exercise A58

(a) The converse is as follows.

If $m - n$ is an even integer,
then m and n are both even integers.

(b) The original statement is true.

Suppose that m and n are both even; then

$$m = 2p, \ n = 2q, \quad \text{where } p, q \text{ are integers.}$$

Then

$$\begin{aligned} m - n &= 2p - 2q \\ &= 2(p - q), \end{aligned}$$

which is even, since $p - q$ is an integer.

The converse is false.

For example, $7 - 3 = 4$ is even, but 7 and 3 are both odd.

Solution to Additional Exercise A59

(a) Let $P(n)$ be the statement

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}.$$

$P(2)$ is true, since

$$\frac{1}{1 \times 2} = \frac{1}{2} = \frac{2-1}{2}.$$

Assume that $P(k)$ is true, that is

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(k-1)k} = \frac{k-1}{k}.$$

We wish to deduce that $P(k+1)$ is true:

$$\begin{aligned} &\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} \\ &\quad = \frac{k}{k+1}. \end{aligned}$$

Now

$$\begin{aligned} &\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} \\ &\quad = \frac{k-1}{k} + \frac{1}{k(k+1)} \quad (\text{by } P(k)) \\ &\quad = \frac{(k-1)(k+1)+1}{k(k+1)} \\ &\quad = \frac{k^2}{k(k+1)} = \frac{k}{k+1}. \end{aligned}$$

Thus, for $k = 2, 3, \ldots,$

$$P(k) \implies P(k+1).$$

Hence, by mathematical induction, $P(n)$ is true for $n = 2, 3, \ldots$.

(b) Let $P(n)$ be the statement

$3^{2n} - 1$ is divisible by 8.

$P(1)$ is true, since

$$3^2 - 1 = 9 - 1 = 8,$$

which is divisible by 8.

Assume that $P(k)$ is true, that is

$3^{2k} - 1$ is divisible by 8.

We wish to deduce that $P(k+1)$ is true:

$3^{2(k+1)} - 1$ is divisible by 8.

Now

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^2 3^{2k} - 1 \\ &= 9 \times 3^{2k} - 1 \\ &= 8 \times 3^{2k} + (3^{2k} - 1), \end{aligned}$$

which is also divisible by 8 by $P(k)$.

Thus, for $k = 1, 2, \ldots,$

$$P(k) \implies P(k+1).$$

Hence, by mathematical induction, $P(n)$ is true for $n = 1, 2, \ldots$.

Solution to Additional Exercise A60

(a) This statement is false.

For example, $-4 < 3$, but $(-4)^2 \not< 3^2$.

(b) This statement is false.

For example, if $x = 1$, then $x^2 - x = 0$, not 2.

(c) This statement is true.

One value of x satisfying $x^2 - x = 2$ is $x = 2$.

(d) This statement is false.

$$\begin{aligned} x^2 - x = -1 &\iff x^2 - x + 1 = 0 \\ &\iff \left(x - \tfrac{1}{2}\right)^2 + \tfrac{3}{4} = 0, \\ &\qquad \text{(by completing the square)} \end{aligned}$$

which is not possible for any real x.
(See Worked Exercise A77 in Unit A4 for a reminder of how to complete the square.)

(e) This statement is false.

For example, if $x = y = 1$, then x/y and y/x are both the integer 1.

(f) This statement is true.

We prove it by mathematical induction.

Let $P(n)$ be the statement

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

$P(1)$ is true, since

$$\tfrac{1}{6} \times 1 \times (1+1)(2+1) = \frac{2 \times 3}{6} = 1 = 1^2.$$

Assume that $P(k)$ is true, that is

$$1^2 + 2^2 + \cdots + k^2 = \tfrac{1}{6}k(k+1)(2k+1).$$

We wish to deduce that $P(k+1)$ is true:

$$\begin{aligned} &1^2 + 2^2 + \cdots + (k+1)^2 \\ &\quad = \tfrac{1}{6}(k+1)(k+2)(2k+3). \end{aligned}$$

Now,

$$\begin{aligned} &1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\ &\quad = \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2 \quad (\text{by } P(k)) \\ &\quad = \tfrac{1}{6}(k+1)(k(2k+1) + 6(k+1)) \\ &\quad = \tfrac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &\quad = \tfrac{1}{6}(k+1)(k+2)(2k+3). \end{aligned}$$

Hence

$$P(k) \implies P(k+1), \text{ for } k \geq 1.$$

Hence, by mathematical induction, $P(n)$ is true for $n = 1, 2, \ldots$.

(g) Let $P(n)$ be the statement

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{2n}.$$

Then $P(2)$ is false, since

$$1 - \tfrac{1}{2} \neq \tfrac{1}{4}.$$

Hence the statement is false.

(In fact, as you can check,

$$P(k) \implies P(k+1), \quad \text{for } k \geq 2;$$

that is, step 2 of a proof by mathematical induction works, even though step 1 does not.

The correct expression for the product is $1/n$.)

Solution to Additional Exercise A61

Suppose that the given statement is false; that is, there are real numbers a and b for which

$$(a+b)^2 < 4ab.$$

Then

$$a^2 + 2ab + b^2 < 4ab,$$

so

$$a^2 - 2ab + b^2 < 0,$$

so

$$(a-b)^2 < 0.$$

But $(a-b)^2$ is a square, so cannot be negative. This is a contradiction, so the given statement must be true.

Hence

$$(a+b)^2 \geq 4ab \quad \text{for all real numbers } a \text{ and } b.$$

Solution to Additional Exercise A62

(a) The contrapositive is as follows.

If n is not divisible by 3,
then n^2 is not divisible by 3.

(b) Suppose that n is not divisible by 3. Then

$$n = 3k+1 \quad \text{or} \quad n = 3k+2,$$

for some integer k.

If $n = 3k+1$, then

$$\begin{aligned} n^2 &= 9k^2 + 6k + 1 \\ &= 3k(3k+2) + 1, \end{aligned}$$

which is not divisible by 3.

If $n = 3k+2$, then

$$\begin{aligned} n^2 &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1, \end{aligned}$$

which is not divisible by 3.

Hence the contrapositive is true.
Hence the original statement is true.

Solution to Additional Exercise A63

We show that properties E1, E2 and E3 hold.

E1 Let $n \in \mathbb{Z}$. Then $n + n = 2n$, which is even since n is an integer, so $n \sim n$. Thus $\sim$ is reflexive.

E2 Let $m, n \in \mathbb{Z}$ and suppose that $m \sim n$. Then $m + n$ is even, that is, $n + m$ is even. Hence $n \sim m$. Thus $\sim$ is symmetric.

E3 Let $l, m, n \in \mathbb{Z}$ and suppose that $l \sim m$ and $m \sim n$. Then $l + m$ is even and $m + n$ is even. Hence

$$l + m = 2j \quad \text{and} \quad m + n = 2k,$$

where $j, k \in \mathbb{Z}$. Hence

$$l = 2j - m \quad \text{and} \quad n = 2k - m,$$

so

$$\begin{aligned} l + n &= 2j - m + 2k - m \\ &= 2(j + k - m), \end{aligned}$$

which is even, since $j + k - m$ is an integer. Hence $l \sim n$. Thus $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric and transitive, it is an equivalence relation.

Solution to Additional Exercise A64

$\sim$ is not reflexive because, for example, $1 \nsim 1$ since $2 \times 1 - 1 = 1$ is not divisible by 7.

$\sim$ is not symmetric because, for example, $5 \sim 3$ since $2 \times 5 - 3 = 7$ which is divisible by 7, but $3 \nsim 5$ since $2 \times 3 - 5 = 1$ which is not divisible by 7.

$\sim$ is not transitive because, for example, $5 \sim 3$ and $3 \sim 6$ since $2 \times 5 - 3 = 7$ and $2 \times 3 - 6 = 0$ which are both divisible by 7, but $5 \nsim 6$ since $2 \times 5 - 6 = 4$ which is not divisible by 7.

Solution to Additional Exercise A65

(a) We show that properties E1, E2 and E3 hold.

E1 For any function $f : \mathbb{R} \to \mathbb{R}$, $f(0) = f(0)$, so $f \sim f$ and hence the relation is reflexive.

E2 If $f \sim g$ so that $f(0) = g(0)$, then $g(0) = f(0)$, so $g \sim f$ and hence the relation is symmetric.

E3 If $f \sim g$ and $g \sim h$ so that $f(0) = g(0)$ and $g(0) = h(0)$, then $f(0) = h(0)$, so $f \sim h$ and hence the relation is transitive.

Therefore this is an equivalence relation.

(b) Each equivalence class consists of all functions in A that take a particular value at 0; that is, each equivalence class is of the form

$$\{f \in A : f(0) = r\},$$

for some $r \in \mathbb{R}$.

A suitable set of representatives is the set of all functions of the form

$$f(x) = r \quad \text{where } r \in \mathbb{R}.$$

(There are many other possibilities for the set of representatives, such as the set of all functions of the form

$$f(x) = x + r \quad \text{where } r \in \mathbb{R},$$

or the set of all functions of the form

$$f(x) = x^2 + r \quad \text{where } r \in \mathbb{R}.)$$

Solution to Additional Exercise A66

We show that properties E1, E2 and E3 hold.

E1 Let $z = x + iy \in \mathbb{C}$. Then

$$x - x = 0 = 5(y - y),$$

so $z \sim z$. Hence the relation is reflexive.

E2 Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be elements of $\mathbb{C}$. Suppose that $z_1 \sim z_2$ so that $x_1 - x_2 = 5(y_1 - y_2)$. Then

$$\begin{aligned} x_2 - x_1 &= -(x_1 - x_2) \\ &= -5(y_1 - y_2) \\ &= 5(y_2 - y_1), \end{aligned}$$

so $z_2 \sim z_1$. Hence the relation is symmetric.

E3 Let $z_3 = x_3 + iy_3$, and suppose that $z_1 \sim z_2$ and $z_2 \sim z_3$ so that

$$x_1 - x_2 = 5(y_1 - y_2)$$

and

$$x_2 - x_3 = 5(y_2 - y_3).$$

Then

$$\begin{aligned} x_1 - x_3 &= x_1 - x_2 + x_2 - x_3 \\ &= 5(y_1 - y_2) + 5(y_2 - y_3) \\ &= 5(y_1 - y_3), \end{aligned}$$

so $z_1 \sim z_3$. Hence the relation is transitive.

Therefore this is an equivalence relation.

Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are related by this relation if

$$x_1 - x_2 = 5(y_1 - y_2);$$

that is, if

$$x_1 - 5y_1 = x_2 - 5y_2.$$

Hence the equivalence classes are the lines $x - 5y = r$, for each real number r; that is, the lines in the complex plane with gradient $\frac{1}{5}$.

Solution to Additional Exercise A67

E1 Let $n \in \mathbb{N}$. Then

$$\frac{n}{n} = 1 = 10^0,$$

so $n \sim n$. Thus $\sim$ is reflexive.

E2 Let $m, n \in \mathbb{N}$ and suppose that $m \sim n$. Then

$$\frac{m}{n} = 10^k$$

for some integer k. Hence

$$\frac{n}{m} = 10^{-k}.$$

Therefore, since $-k$ is an integer, $n \sim m$. Thus $\sim$ is symmetric.

E3 Let $l, m, n \in \mathbb{N}$ and suppose that $l \sim m$ and $m \sim n$. Then

$$\frac{l}{m} = 10^j \quad \text{and} \quad \frac{m}{n} = 10^k$$

for some integers j and k. Hence

$$\frac{l}{n} = \frac{l}{m}\frac{m}{n} = 10^j 10^k = 10^{j+k}.$$

Therefore, since $j + k$ is an integer, $l \sim n$. Thus $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric and transitive, it is an equivalence relation.

Solution to Additional Exercise A68

(a) We start by finding a particular equivalence class. We have, for example,

$$\begin{aligned}
[\![2]\!] &= \{n \in \mathbb{N} : 2 \sim n\} \\
&= \left\{n \in \mathbb{N} : \frac{2}{n} = 10^k \text{ for some integer } k\right\} \\
&= \left\{n \in \mathbb{N} : n = \frac{2}{10^k} \text{ for some integer } k\right\} \\
&= \left\{n \in \mathbb{N} : n = 2 \times 10^k \text{ for some integer } k\right\} \\
&\quad \text{(because as } k \text{ ranges over all integers,} \\
&\quad \text{so does } -k \text{, and } 2/10^{-k} = 2 \times 10^k) \\
&= \left\{2 \times 10^k : k \in \{0, 1, 2, \dots\}\right\} \\
&= \{2, 20, 200, 2000, \dots\}.
\end{aligned}$$

By a similar argument, for any natural number m we have

$$\begin{aligned}
[\![m]\!] &= \left\{m \times 10^k : k \in \{0, 1, 2, \dots\}\right\} \\
&= \{m, 10m, 100m, 1000m, \dots\}.
\end{aligned}$$

A set of representatives for the equivalence classes of $\sim$ is the set of all natural numbers whose last digit is not zero.

The equivalence classes of $\sim$ are all the sets of the form

$$[\![m]\!] = \{m, 10m, 100m, 1000m, \dots\},$$

where m is a positive integer whose last digit is not zero.

(b) In Worked Exercise A75 in Unit A3 it was found that

$$\begin{aligned}
[\![3.7]\!] &= \{k + 3.7 : k \in \mathbb{Z}\} \\
&= \{k + 0.7 : k \in \mathbb{Z}\}.
\end{aligned}$$

This set can also be written (less concisely) as

$$\begin{aligned}
[\![3.7]\!] = \{\dots, -2.3,\ -1.3,\ -0.3,& \\
0.7,\ 1.7,\ 2.7,\ 3.7, \dots\}&.
\end{aligned}$$

In general, for any real number x, we have

$$\begin{aligned}
[\![x]\!] &= \{k + x : k \in \mathbb{Z}\} \\
&= \{k + p : k \in \mathbb{Z}\},
\end{aligned}$$

where p is the number obtained by subtracting the integer part of x from x; that is, $p = x - \lfloor x \rfloor$. This set can also be written as

$$\begin{aligned}
\{\dots, -3 + p,\ -2 + p,\ -1 + p,& \\
p,\ 1 + p,\ 2 + p,\ 3 + p,\ \dots\}&.
\end{aligned}$$

A set of representatives for the equivalence classes of $\sim$ is the interval $[0, 1)$, since every equivalence class has exactly one element in this interval.

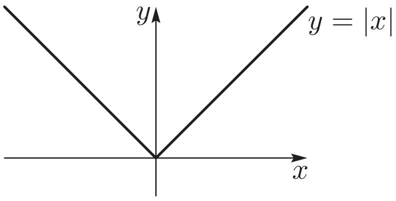
The equivalence classes of $\sim$ are all the sets of the form $\{k + p : k \in \mathbb{Z}\}$, where p is a number in the interval $[0, 1)$.

Additional exercises for Unit A4

Section 1

Additional Exercise A69

The graph of $y = |x|$ is shown below.



Sketch the graphs of the following.

(a) $y = |\frac{x}{2}|$ (b) $y = |\frac{x}{2} - 2|$

(c) $y = -|x| - 1$

Section 2

Additional Exercise A70

Sketch the graph of $f(x) = \frac{1}{5}x^5 - x^3$

Additional Exercise A71

Sketch the graph of $f(x) = \dfrac{4x+3}{x-7}$

Additional Exercise A72

Sketch the graph of the linear rational function

$$f(x) = \frac{4x+1}{3x-5}.$$

Additional Exercise A73

Sketch the graph of $f(x) = \dfrac{2x}{x^2+x-2}$

Additional Exercise A74 Challenging

Sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

Hint: To simplify after differentiating, multiply the numerator and denominator by $(x^2+1)^{1/2}$.

Section 3

Additional Exercise A75

Sketch the graph of $f(x) = \sqrt{x}\cos x$.

Additional Exercise A76 Challenging

Sketch the graph of $f(x) = 2\cos x - x$.

Hint: In step 3 find an interval in which there is an x-intercept, and omit step 4.

Additional Exercise A77 Challenging

Sketch the graph of $f(x) = \dfrac{\cos x}{x}$.

Additional Exercise A78

Sketch the graph of each of the following hybrid functions.

(a) $f(x) = \begin{cases} |x|, & x \leq 1 \\ -x, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} 2-x, & x < -1 \\ x^2-1, & -1 \leq x \leq 1 \\ \log x, & x > 1 \end{cases}$

Section 4

Additional Exercise A79

Let $f(x) = \tanh x$.

(a) Show that f is an odd function.

(b) Show that

$$f(x) = \frac{1-e^{-2x}}{1+e^{-2x}}.$$

(c) Show that $f'(x) = \operatorname{sech}^2 x$, and deduce that $f'(x) > 0$ for all x in $\mathbb{R}$.

Additional Exercise A80

Prove that

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

Additional Exercise A81

Use the results of Additional Exercise A79 to sketch the graph of the function

$$f(x) = \tanh x.$$

Additional Exercise A82

Sketch the graph of the function $f(x) = \coth x$.

Section 5

Additional Exercise A83

Determine the centre and radius of the circle given by the following equation.

$$3x^2 + 3y^2 - 12x - 48y = 0$$

Additional Exercise A84

Identify the curves described by the following parametric equations.

(a) $x = t, \quad y = 1/t$.

(b) $x = t - 1, \quad y = 4 - 3t$.

(c) $x = 2t, \quad y = 1 + 3t^2$.

Additional Exercise A85

Show that the points given by the parametrisation

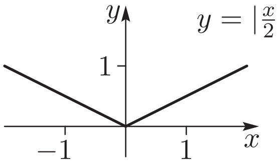
$$f(t) = (pt^3, qt^3)$$

all lie on the line through the points $(0, 0)$ and (p, q), where $p \neq 0$.

Solutions to additional exercises for Unit A4

Solution to Additional Exercise A69

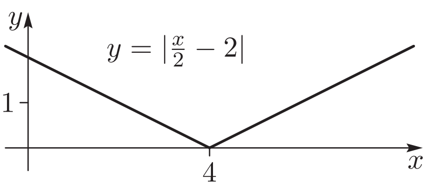
(a) The graph of $y = |\frac{x}{2}|$ is obtained from the graph of $y = |x|$ by a $(2, 1)$-scaling.



(b) Consider $y = |x|$, then we replace x by $x/2$ to obtain $y = |\frac{x}{2}|$. We then replace x by $x - 4$ to obtain the equation

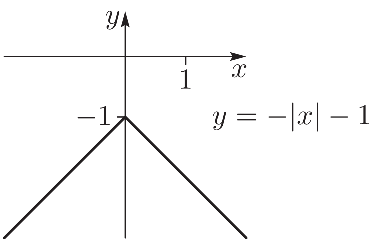
$$\begin{aligned} y &= \left| \frac{x-4}{2} \right| \\ &= \left| \frac{x}{2} - 2 \right|. \end{aligned}$$

So the graph of $y = |\frac{x}{2} - 2|$ is obtained from the graph of $y = |x|$ by a $(2, 1)$-scaling and a $(4, 0)$-translation.



(c) Consider $y = |x|$, then we multiply the right-hand side by -1 to obtain the equation $y = -|x|$, then we add -1 to the right-hand side to obtain $y = -|x| - 1$.

So the graph of $y = -|x| - 1$ is obtained from the graph of $y = |x|$ by a $(1, -1)$-scaling and a $(0, -1)$-translation.



Solution to Additional Exercise A70

$f(x) = \frac{1}{5}x^5 - x^3$.

1. The domain of f is $\mathbb{R}$.
2. f is odd, since, for all x in $\mathbb{R}$,

$$\begin{aligned} f(-x) &= \tfrac{1}{5}(-x)^5 - (-x)^3 \\ &= -(\tfrac{1}{5}x^5 - x^3) = -f(x). \end{aligned}$$

It is therefore sufficient to consider the features of the graph of f for $x \geq 0$, and then to rotate the graph we obtain about the origin.

3. $$\begin{aligned} f(x) &= \tfrac{1}{5}x^5 - x^3 \\ &= x^3(\tfrac{1}{5}x^2 - 1), \end{aligned}$$

so $f(x) = 0$ when $x = 0$ and $x = \pm\sqrt{5}$.

So the x-intercepts are 0 and $\pm\sqrt{5}$, and the y-intercept is 0.

4. We construct a table of signs for f, for $x > 0$.

x	0	$(0, \sqrt{5})$	$\sqrt{5}$	$(\sqrt{5}, \infty)$
x^3	$+$	$+$	$+$	$+$
$\frac{1}{5}x^2 - 1$	$-$	$-$	0	$+$
$f(x)$	0	$-$	0	$+$

Thus

- f is positive on the interval $(\sqrt{5}, \infty)$;
- f is negative on the interval $(0, \sqrt{5})$.

5. $f'(x) = x^4 - 3x^2 = x^2(x^2 - 3)$, so

$$\begin{aligned} f'(x) &= 0 \quad \text{when } x = 0 \text{ and } \pm\sqrt{3}; \\ f'(x) &> 0 \quad \text{when } x^2 > 3; \\ f'(x) &< 0 \quad \text{when } x^2 < 3. \end{aligned}$$

Thus, for $x > 0$,

- f is increasing on the interval $(\sqrt{3}, \infty)$;
- f is decreasing on the interval $(0, \sqrt{3})$;
- f has a stationary point at $\sqrt{3}$.

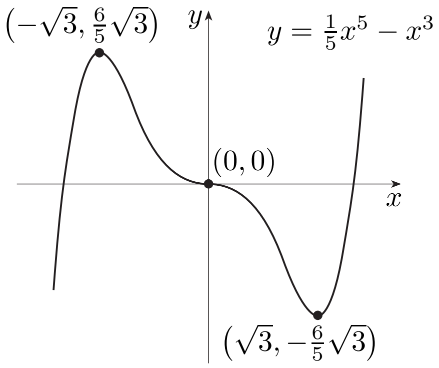
By the First Derivative Test, we deduce that, for $x > 0$,

- there is a local minimum at $x = \sqrt{3}$ with $f(\sqrt{3}) = -\frac{6}{5}\sqrt{3}$;
- there is a horizontal point of inflection at $x = 0$.

6. The power of x in the dominant term is odd, and the dominant term has a plus sign so

$$f(x) \to \infty \quad \text{as } x \to \infty.$$

This information enables us to sketch the graph.



Solution to Additional Exercise A71

$$f(x) = \frac{4x+3}{x-7}.$$

1. The domain of f is $\mathbb{R}$, excluding 7.
2. f is neither even nor odd, since its domain is not symmetric about the origin.
3. $f(x) = 0$ when $x = -\frac{3}{4}$, so the x-intercept is $-\frac{3}{4}$.

   The y-intercept is $f(0) = -\frac{3}{7}$.
4. We construct a table of signs for f.

x	$(-\infty, -\frac{3}{4})$	$-\frac{3}{4}$	$(-\frac{3}{4}, 7)$	7	$(7, \infty)$
$4x+3$	$-$	0	$+$	$+$	$+$
$x-7$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	$*$	$+$

   Thus
   - f is positive on the intervals $(-\infty, -\frac{3}{4})$ and $(7, \infty)$;
   - f is negative on the interval $(-\frac{3}{4}, 7)$.
5. Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x-7)4 - (4x+3)}{(x-7)^2} \\ &= \frac{-31}{(x-7)^2}, \end{aligned}$$

   so $f'(x) < 0$ for all x in the domain.

   Thus f is decreasing on each interval of its domain.
6. The denominator is 0 when $x = 7$, so the line $x = 7$ is a vertical asymptote.

   Also, from step 4,

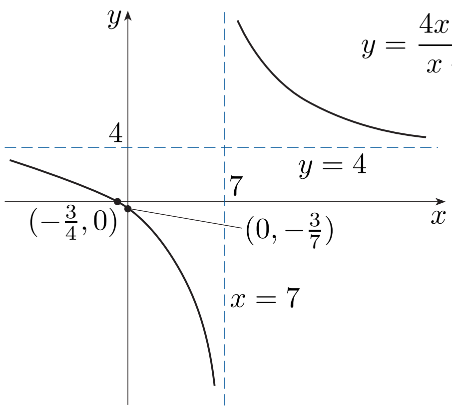
$$\begin{aligned} f(x) &\to -\infty \quad \text{as } x \to 7^-, \\ f(x) &\to \infty \quad \text{as } x \to 7^+. \end{aligned}$$

   The dominant term of the numerator is $4x$ and the dominant term of the denominator is x, so we consider the simpler function

$$g(x) = \frac{4x}{x} = 4.$$

   Therefore the line $y = 4$ is a horizontal asymptote.

This information enables us to sketch the graph.



Solution to Additional Exercise A72

$$f(x) = \frac{4x+1}{3x-5}.$$

1. The domain of f is $\mathbb{R} - \{\frac{5}{3}\}$.
2. f is neither even nor odd, since its domain is not symmetric about the origin.
3. We have $f(x) = 0$ when $x = -\frac{1}{4}$, so the x-intercept is $-\frac{1}{4}$. The y-intercept is $f(0) = -\frac{1}{5}$.
4. We construct a table of signs for f.

x	$(-\infty, -\frac{1}{4})$	$-\frac{1}{4}$	$(-\frac{1}{4}, \frac{5}{3})$	$\frac{5}{3}$	$(\frac{5}{3}, \infty)$
$4x+1$	$-$	0	$+$	$+$	$+$
$3x-5$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	$*$	$+$

   Thus
   - f is positive on the intervals $(-\infty, -\frac{1}{4})$ and $(\frac{5}{3}, \infty)$;
   - f is negative on the interval $(-\frac{1}{4}, \frac{5}{3})$.
5. Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(3x-5)4 - (4x+1)3}{(3x-5)^2} \\ &= \frac{-23}{(3x-5)^2}, \end{aligned}$$

   so $f'(x) < 0$ for all x in the domain; that is, f is decreasing on each interval of its domain.

6. The denominator is 0 when $x = \frac{5}{3}$, so the line $x = \frac{5}{3}$ is a vertical asymptote.

   Also, by step 4,

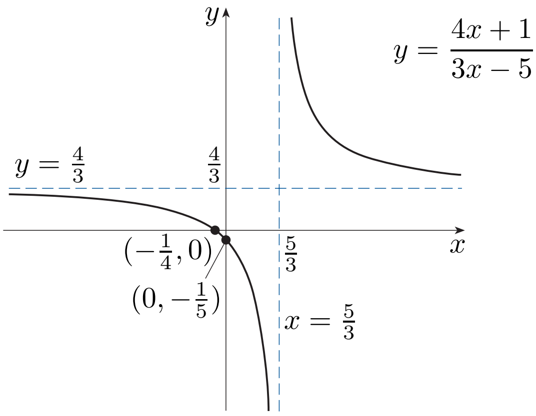
$$f(x) \to -\infty \quad \text{as } x \to \tfrac{5}{3}^-,$$
$$f(x) \to \infty \quad \text{as } x \to \tfrac{5}{3}^+.$$

   The dominant term of the numerator is $4x$ and the dominant term of the denominator is $3x$, so we consider the simpler function

$$g(x) = \frac{4x}{3x} = \tfrac{4}{3}.$$

   Therefore the line $y = \frac{4}{3}$ is a horizontal asymptote.

This information enables us to sketch the graph.



Solution to Additional Exercise A73

$$f(x) = \frac{2x}{x^2 + x - 2}.$$

1. We factorise $f(x)$ as follows:

$$f(x) = \frac{2x}{(x-1)(x+2)}.$$

   Thus the domain of f is $\mathbb{R} - \{1, -2\}$.

2. f is neither even nor odd, since its domain is not symmetric about the origin.

3. $f(x) = 0$ when $x = 0$, so 0 is both the x-intercept and the y-intercept.

4. We construct a table of signs for f.

x		-2	$(-2,0)$	0	$(0,1)$	1	
$2x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	$*$	$+$	0	$-$	$*$	$+$

   Thus

   - f is positive on the intervals $(-2, 0)$ and $(1, \infty)$;
   - f is negative on the intervals $(-\infty, -2)$ and $(0, 1)$.

5. Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^2+x-2)2 - 2x(2x+1)}{(x^2+x-2)^2} \\ &= \frac{-2x^2-4}{(x^2+x-2)^2} \\ &= \frac{-2(x^2+2)}{(x^2+x-2)^2}, \end{aligned}$$

   so $f'(x) < 0$ for all x in the domain.

   Thus f is decreasing on each interval of its domain.

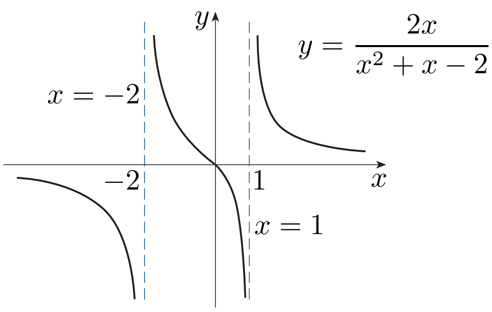
6. The denominator is 0 when $x = 1$ and $x = -2$, so the lines $x = -2$ and $x = 1$ are vertical asymptotes.

   Also, from step 4,

$$f(x) \to -\infty \quad \text{as } x \to -2^-,$$
$$f(x) \to \infty \quad \text{as } x \to -2^+,$$
$$f(x) \to -\infty \quad \text{as } x \to 1^-,$$
$$f(x) \to \infty \quad \text{as } x \to 1^+.$$

   The dominant term of the numerator is $2x$ which has a lower power of x than the dominant term of the denominator which is x^2, so the line $y = 0$ is a horizontal asymptote.

This information enables us to sketch the graph.

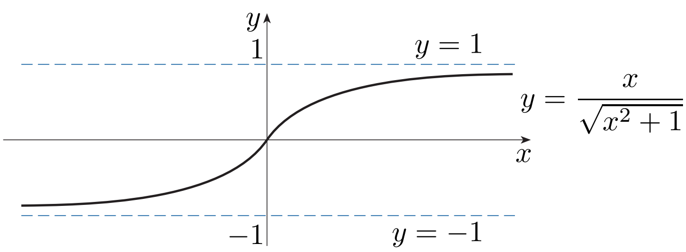


Solution to Additional Exercise A74

$f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

1. The domain of f is $\mathbb{R}$, since $x^2+1>0$ for all x in $\mathbb{R}$.
2. f is odd, since for all x in $\mathbb{R}$,
$$\begin{aligned} f(-x) &= \frac{-x}{\sqrt{(-x)^2+1}} \\ &= \frac{-x}{\sqrt{x^2+1}} = -f(x). \end{aligned}$$
It is therefore sufficient to consider the features of the graph of f for $x \geq 0$, and then to rotate the graph we obtain about the origin.
3. The solution of $f(x)=0$, is $x=0$, so the x-intercept and the y-intercept are both 0.
4. Since $f(x)$ has the same sign as x, f is positive on the interval $(0,\infty)$.
5. Using the quotient rule,
$$\begin{aligned} f'(x) &= \frac{\sqrt{x^2+1} - x(\frac{1}{2}(x^2+1)^{-1/2}2x)}{x^2+1} \\ &= \frac{x^2+1-x^2}{(x^2+1)^{3/2}} \\ &= \frac{1}{(x^2+1)^{3/2}}. \end{aligned}$$
So $f'(x)>0$ for all x in $\mathbb{R}$; that is, f is increasing on $\mathbb{R}$.
(In addition, $f'(0)=1$, so the graph has gradient 1 at the origin – this is not part of the strategy, but is helpful here.)
6. As x becomes large and positive the function behaves in a similar way to the simpler function $g(x)=x/x=1$. Therefore the line $y=1$ is a horizontal asymptote and
$$f(x) \to 1 \quad \text{as } x \to \infty.$$
Since f is odd, the line $y=-1$ is also a horizontal asymptote and
$$f(x) \to -1 \quad \text{as } x \to -\infty.$$

This information enables us to sketch the graph.

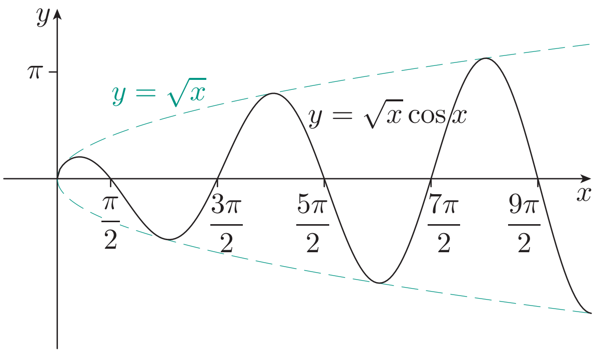


Solution to Additional Exercise A75

$f(x) = \sqrt{x}\cos x$.

1. The function f has domain $[0,\infty)$.
2. f is neither even nor odd, since its domain is not symmetrical about the origin.
3. We have $f(x)=0$ when $x=0$ and when $\cos x = 0$.
So the x-intercepts are $0, \pi/2, 3\pi/2, \ldots$.
The y-intercept is 0 since $f(0)=0$.
4. The intervals on which f is positive or negative (for $x>0$) alternate between the x-intercepts in the same way as for the cosine function.
For $x>0$,
   - f is positive on $(0,\pi/2), (3\pi/2, 5\pi/2), (7\pi/2, 9\pi/2), \ldots$,
   - f is negative on $(\pi/2, 3\pi/2), (5\pi/2, 7\pi/2), \ldots$.
5. We omit this step as it is unlikely that $f'(x)=0$ will be easy to solve.
6. The function has no asymptotes.
7. Since $-1 \leq \cos x \leq 1$ for all real numbers x, we have
$$-\sqrt{x} \leq \sqrt{x}\cos x \leq \sqrt{x} \quad \text{for } x>0.$$
That is,
$$-\sqrt{x} \leq f(x) \leq \sqrt{x} \quad \text{for } x>0,$$
so, the graph of f lies between the graphs $y=\sqrt{x}$ and $y=-\sqrt{x}$. These graphs are the construction lines for this function.
The function f has the following features:
$$f(x) = \sqrt{x} \text{ when } \cos x = 1;$$
$$f(x) = -\sqrt{x} \text{ when } \cos x = -1.$$
The graph of f
   - meets the construction line $y=\sqrt{x}$ when $x = 0, 2\pi, 4\pi, \ldots$;
   - meets the construction line $y=-\sqrt{x}$ when $x=\pi, 3\pi, \ldots$.

This information enables us to sketch the graph.



Solution to Additional Exercise A76

$f(x) = 2\cos x - x$.

1. The domain of f is $\mathbb{R}$.
2. f is not even, odd or periodic.
3. Although $f(x) = 0$ is not easy to solve, we know that $f(0) = 2$, which is positive, and $f(\pi/2) = -\pi/2$, which is negative, so there is an x-intercept in the interval $(0, \pi/2)$.

   The y-intercept is 2.
4. We omit this step.
5. $f'(x) = -2\sin x - 1$, so $f'(x) = 0$ when $\sin x = -\frac{1}{2}$; that is, when $x = -\pi/6 + 2k\pi$ or $x = -5\pi/6 + 2k\pi$, for any integer k. These are stationary points.

   At these points, we know that f' either changes from positive to negative, or negative to positive, so by the First Derivative Test, we deduce that these stationary points alternate between maxima and minima.
6. The function has no asymptotes.
7. Since

   $$-2 \le 2\cos x \le 2, \quad \text{for all } x \text{ in } \mathbb{R},$$

   then, for all x in $\mathbb{R}$

   $$-2 - x \le 2\cos x - x \le 2 - x.$$

   Therefore the graph of f lies between the construction lines $y = -2 - x$ and $y = 2 - x$.

   We have

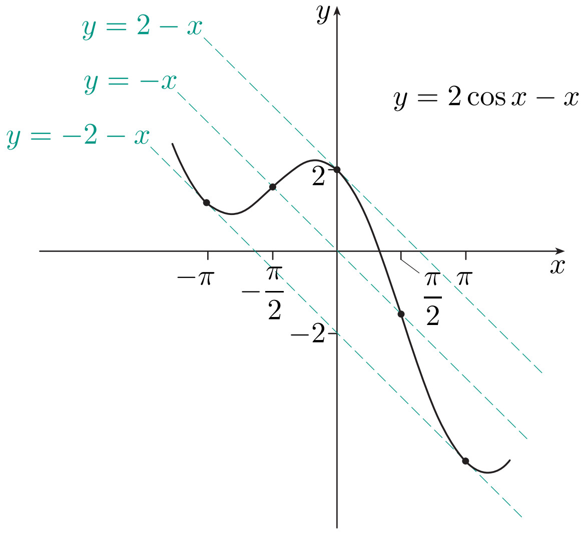
   $$f(x) = 2 - x \text{ when } \cos x = 1$$
   $$f(x) = -2 - x \text{ when } \cos x = -1$$
   $$f(x) = -x \text{ when } \cos x = 0.$$

   So, the graph of f

   - meets the construction line $y = 2 - x$ when $x = 2k\pi$;
   - meets the construction line $y = -2 - x$ when $x = (2k + 1)\pi$;
   - meets the line $y = -x$ when $x = (k + \frac{1}{2})\pi$;

   for any integer k.

This information enables us to sketch the graph.



Solution to Additional Exercise A77

$f(x) = \dfrac{\cos x}{x}$.

1. The domain of f is $\mathbb{R}$, excluding 0.
2. f is odd, since, for all x in the domain,

   $$f(-x) = \frac{\cos(-x)}{-x} = \frac{\cos x}{-x} = -\frac{\cos x}{x} = -f(x).$$

   We consider the features of the graph of f for $x \ge 0$, and then rotate the graph we obtain about the origin.
3. $f(x) = 0$ whenever $\cos x = 0$; that is, $f(x) = 0$ when $x = (k + \frac{1}{2})\pi$, for any integer k.

   $f(0)$ is not defined, so there is no y-intercept.
5. The equation $f'(x) = 0$ is not easy to solve, so we omit this step.
6. The denominator is 0 when $x = 0$, so the line $x = 0$ is a vertical asymptote, and

   $$f(x) \to \infty \quad \text{as } x \to 0^+,$$
7. We know that $-1 \le \cos x \le 1$ for all x in $\mathbb{R}$.

For $x > 0$, we have $\dfrac{1}{x} > 0$, so

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}.$$

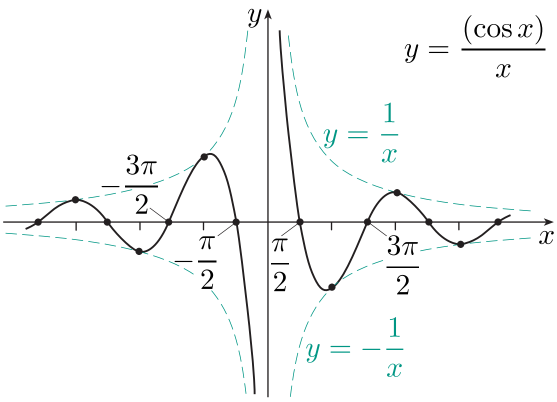
So the graph of f lies between the construction lines $y = -1/x$ and $y = 1/x$, and meets these when $\cos x = \pm 1$.

So, for $x > 0$, the graph of f

- meets the construction line $y = 1/x$ when $x = 2k\pi$;
- meets the construction line $y = -1/x$ when $x = (2k+1)\pi$;

for any positive integer k.

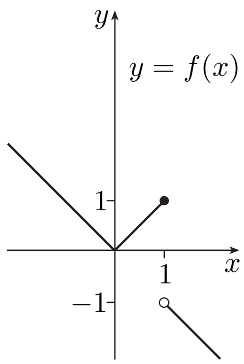
This information enables us to sketch the graph.



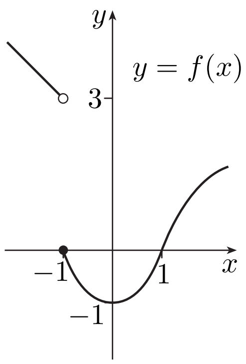
Solution to Additional Exercise A78

Each of these functions has domain $\mathbb{R}$.

(a)



(b)



Solution to Additional Exercise A79

(a)
$$\begin{aligned} f(-x) &= \tanh(-x) \\ &= \frac{\sinh(-x)}{\cosh(-x)} \\ &= \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}} \\ &= -\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right) \\ &= -\tanh x = -f(x), \end{aligned}$$

so tanh is an odd function.

(b) $f(x) = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}.$

Dividing both numerator and denominator by e^x (non-zero for all x in $\mathbb{R}$), we obtain

$$f(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$

(c) $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x}.$

Differentiating the quotient, we obtain

$$\begin{aligned} f'(x) &= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x. \end{aligned}$$

Since $\operatorname{sech}^2 x$ is positive for all x in $\mathbb{R}$, it follows that

$$f'(x) > 0, \quad \text{for all } x \text{ in } \mathbb{R}.$$

Solution to Additional Exercise A80

$$\begin{aligned} \tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)} \\ &= \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}. \end{aligned}$$

Dividing both numerator and denominator by $\cosh x \cosh y$, we obtain

$$\begin{aligned} \tanh(x+y) &= \frac{\dfrac{\sinh x}{\cosh x} + \dfrac{\sinh y}{\cosh y}}{1 + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}} \\ &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}. \end{aligned}$$

Solution to Additional Exercise A81

$f(x) = \tanh x = \dfrac{\sinh x}{\cosh x}.$

1. $\tanh x$ has domain $\mathbb{R}$, since $\cosh x$ never takes the value 0.
2. $\tanh x$ is odd, by Exercise A79(a). It is therefore sufficient to consider the features of the graph of f for $x \geq 0$, and then to rotate the graph we obtain through π about the origin.

3. We know that $\cosh x \geq 1$ for all x in $\mathbb{R}$, so the x-intercepts of $\tanh x$ are the same as those of $\sinh x$, namely 0.

   So 0 is both the x-intercept and the y-intercept.

4. We know that on the interval $(0, \infty)$, $\tanh x$ is positive because $\sinh x$ is positive.

5. We know, from Exercise A79(c), that $f'(x) > 0$ for all x in $\mathbb{R}$, so there are no stationary points.

   (Since $f'(0) = \operatorname{sech}^2(0) = 1$, the graph of $\tanh x$ has gradient 1 at the origin.)

6. Since $\cosh x \geq 1$, the denominator is never zero, so there are no vertical asymptotes.

   Since $e^{-x} \to 0$ as $x \to \infty$, we know $e^{-2x} \to 0$ as $x \to \infty$. We know from Exercise A79(b) that

$$\tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}},$$

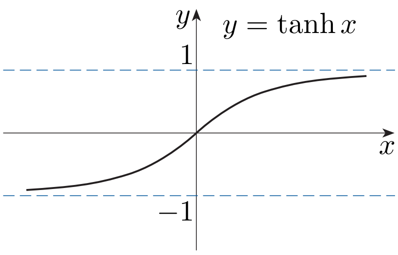
   so as x becomes large and positive the function behaves in a similar way to the simpler function $g(x) = 1/1 = 1$. Therefore the line $y = 1$ is a horizontal asymptote and

$$\tanh x \to 1 \quad \text{as } x \to \infty.$$

   Since the function is odd, the line $y = -1$ is also a horizontal asymptote and

$$\tanh x \to -1 \quad \text{as } x \to -\infty.$$

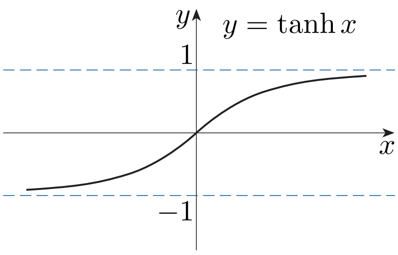
This information enables us to sketch the graph.



Solution to Additional Exercise A82

$f(x) = \coth x = 1/\tanh x$.

We use the graph of $\tanh x$.



1. $\tanh x = 0$ when $x = 0$, so the domain of $\coth x$ is $\mathbb{R}$ excluding 0.

2. $\tanh x$ is odd, so $\coth x$ is odd.

   We consider the features of the graph of f for $x \geq 0$, and then rotate the graph we obtain about the origin.

3. $\coth x$ has neither x- nor y-intercepts.

4. We know that $\tanh x > 0$ when $x > 0$, so $\coth x > 0$ when $x > 0$.

5. $\tanh x$ is increasing on $\mathbb{R}$, so $\coth x$ is decreasing for $x > 0$, and so has no local maxima or minima.

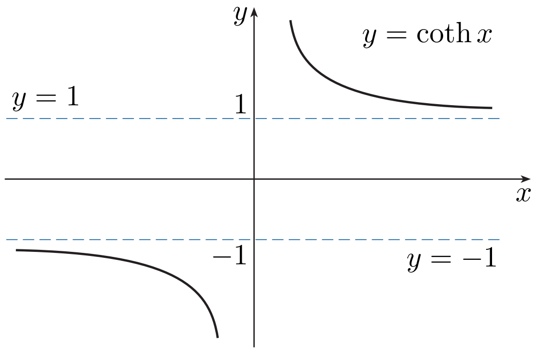
6. $\tanh x \to 1$ as $x \to \infty$, so

$$\coth x \to 1 \quad \text{as } x \to \infty.$$

   $\tanh x$ is small when x is close to 0, so the line $x = 0$ is a vertical asymptote. Also,

$$\coth x \to \infty \quad \text{as } x \to 0^+.$$

This information enables us to sketch the graph.



Solution to Additional Exercise A83

Here the coefficients of x^2 and y^2 are both 3, so we divide the equation by 3 to obtain

$$x^2 + y^2 - 4x - 16y = 0.$$

Completing the square gives

$$(x-2)^2 - 4 + (y-8)^2 - 64 = 0,$$

or

$$(x-2)^2 + (y-8)^2 = 68.$$

So the circle has centre $(2, 8)$ and radius $\sqrt{68} = 2\sqrt{17}$.

Solution to Additional Exercise A84

(a) $y = 1/x$. Hence the curve is the graph of the reciprocal function.

(b) $t = x + 1$, so $y = 4 - 3(x + 1) = 1 - 3x$. Hence the curve is a straight line with gradient -3 and y-intercept 1.

(c) $t = \dfrac{x}{2}$, so $y = 1 + 3\left(\dfrac{x}{2}\right)^2 = 1 + \dfrac{3x^2}{4}$.
Hence the curve is a parabola, symmetric about the y-axis, with vertex at $(0, 1)$.

Solution to Additional Exercise A85

Eliminating t, we obtain

$$t^3 = \frac{x}{p},$$

so

$$y = qt^3 = \frac{q}{p}x,$$

which is the equation of the line through the points $(0, 0)$ and (p, q).